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Désintégrations des noyaux légers de l'émulsion nucléaire par des mésons π^- lents.

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Summary. — Ilford G.5 plates were exposed to slow π^--mesons produced by the Rochester Cyclotron. About 1500 σ-stars were classified according to the number of prongs and according that they were produced in a light nucleus (C, N, O) or a heavy nucleus (Br, Ag) of the emulsion. The examination of stars produced in the light nuclei has allowed to investigate the modes of desintegration of these nuclei. An extension of the scheme of Clark and Ruddlesden is proposed which gives an account of about $\frac{2}{3}$ of the observed stars. The remaining stars contain at least one fragment ($Z > 2$) and seem better interpreted by a mechanism of local heating.

1. – Introduction.

L'étude des désintégrations induites par des mésons π^- lents dans les noyaux légers de l'émulsion nucléaire (carbone, azote, oxygène) a déjà fait l'objet de plusieurs travaux expérimentaux ([1-4]) qui ont mis en évidence leurs princi-

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([1]) D. H. PERKINS: *Phil. Mag.*, **40**, 601 (1949).

([2]) M. G. K. MENON, H. MUIRHEAD et O. ROCHAT: *Phil. Mag.*, **41**, 583 (1950).

([3]) S. ENOMOTO, Y. FUJIMOTO, S. HORIE et Y. TSUZUKI: *Progr. Theor. Phys.*, **7**, 353 (1952).

([4]) V. DE SABBATA, E. MANARESI et G. PUPPI: *Nuovo Cimento*, **10**, 1704 (1953).

pales caractéristiques. Les désintégrations de l'hélium, du carbone et de l'azote ont aussi été étudiées au moyen d'une chambre à diffusion par AMMIRAJU *et al.* (5). Toutefois, certains désaccords existent entre les résultats publiés.

Dans la première partie du présent travail, nous cherchons à préciser les critères permettant de discriminer les étoiles produites dans les noyaux légers des étoiles produites dans les noyaux lourds de l'émulsion nucléaire et nous appliquons ces critères à un millier d'étoiles. Nous obtenons en même temps des résultats sur les fréquences relatives des mésons ρ et σ ainsi que sur la distribution de l'ensemble des étoiles σ d'après le nombre de branches. Nous comparons ces résultats à ceux obtenus par d'autres auteurs.

Dans la deuxième partie du travail, nous examinons dans quelle mesure ces désintégrations peuvent être interprétées au moyen du schéma proposé par CLARK et RUDDLESDEN (6). Ces auteurs ont traité le problème de la désintégration des noyaux du type ${}^{4k}_{2k}X$ (comme ${}^{12}_{6}C$ et ${}^{16}_{8}O$) en les supposant constitués de k particules α et en admettant qu'une seule de ces particules est détruite par l'absorption du méson π^-. Nous étendons cette étude aux noyaux du type ${}^{4k+2}_{2k+1}X$ (comme ${}^{14}_{7}N$) en les supposant constitués de k particules α et d'un deuton, et en admettant que le méson peut interagir soit avec une seule des particules α, soit avec le deuton.

La comparaison de nos résultats expérimentaux avec les prévisions que l'on peut faire d'après le schéma de Clark et Ruddlesden nous amène à proposer une extension de ce schéma, permettant de rendre compte des $\frac{2}{3}$ des désintégrations.

2. – Matériel Expérimental.

Des plaques Ilford G5 de 1000 μm d'épaisseur ont été exposées à des mésons π^- de 18 et de 36 MeV, produits par le synchrocyclotron de Rochester. La proportion de méson μ^- était de l'ordre de 1% (*).

Les plaques ont été sous-développées dans le but de faciliter la discrimination entre les traces de différentes particules. Toutefois, des fluctuations locales de densité de grains rendent incertaine la discrimination entre les traces courtes (< 50 μm). La densité de grains moyenne au minimum d'ionisation est d'environ 25 grains/100 μm.

Pour faciliter le repérage des événements, les plaques ont été recouvertes après développement d'un quadrillage photographique sur film mince (~ 90 μm),

(5) P. AMMIRAJU, M. RINEHART, K. C. ROGERS et L. M. LEDERMAN: *Bull. Am. Phys. Soc.*, **30**/1, 49 (1955).

(6) A. C. CLARK et S. N. RUDDLESDEN: *Proc. Phys. Soc.*, A **64**, 1064 (1951).

(*) Valeur communiquée par le Professeur M. F. KAPLON.

composé de cases de 250 μm de côté numérotées individuellement suivant les lignes et les colonnes [7].

3. – Classement des évènements.

3·1. *Classification des traces.* – Branches. Pour le classement des branches d'étoiles, nous avons repris la classification généralement adoptée:

a) *protons* (p), comprenant protons, deutons et tritons.

b) *particules alpha* (α),

c) *fragments nucléaires* (F), comprenant des noyaux légers de nombre atomique $3 \leqslant Z \leqslant 7$.

Pour les traces de longueur supérieure à 100 μm, la discrimination peut être faite aisément d'après un simple examen visuel de l'ionisation et du scattering en fonction du parcours.

Pour les traces plus courtes que 100 μm, la discrimination devient difficile ou impossible, tant d'après la diffusion multiple (scattering) que d'après l'ionisation. Toutefois, les indéterminations ont souvent pu être levées entre les protons, d'une part, et les particules α et fragments, d'autre part, en tenant compte de conditions de compatibilité entre le nombre, la nature et l'énergie des branches d'une étoile, selon que celle-ci a été produite dans un noyau léger ou dans un noyau lourd. Ces conditions résultent des critères de classement des étoiles qui seront énoncés à la Sect. 3·4.

Reculs. Les noyaux de recul résultant de l'absorption de mésons π^- par les noyaux lourds de l'émulsion ne sont pas classés comme branches mais comme *reculs*.

Certains auteurs ont adopté pour la longueur des reculs une limite supérieure de 5 μm [3,4] ou de 6 μm [8,9]. Cependant, MENON *et al.* [2] ont montré que l'énergie cinétique maximum pouvant être cédée à un recul provenant d'un noyau initial de nombre de masse $A = 100$ est de 6 MeV, dans le cas limite où 9 nucléons sont émis simultanément en sens opposé à celui du recul. Le parcours correspondant à cette énergie dans l'émulsion nucléaire est de l'ordre de 3 μm. Si on tient compte du fait que le nombre moyen de nucléons émis par les noyaux lourds après absorption d'un meson π^- est seulement de l'ordre de 3 ou 4 et qu'ils ne sont généralement émis ni simultanément

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(7) G. VANDERHAEGHE: *CERN*, BS/1 (1954).

(8) W. CHESTON et L. GOLDFARB: *Phys. Rev.*, **78**, 683 (1950).

(9) M. BENEVENTANO, D. CARLSON-LEE, G. STOPPINI, G. BERNARDINI et E. L. GOLDWASSER: *Nuovo Cimento*, **12**, 156 (1954).

ni dans la même direction, on voit que la probabilité d'un recul de plus que 1 à 2 μm doit être très petite. C'est pourquoi nous n'avons considéré comme reculs que les traces noires de longueur inférieure à 2 μm, dans les cas où ces traces n'étaient pas associées à des branches permettant d'identifier les noyaux initiaux comme noyaux légers d'après les critères de la Sect. 3·4.

3·2. *Mésons σ et mésons ρ.* – Une partie des plaques a été explorée en notant tous les mésons s'arrêtant dans l'émulsion. Nous avons classé comme mésons σ ceux dont la fin de la trace est associée à au moins une branche. Nous avons classé comme mésons ρ ceux dont la fin de la trace n'est associée à aucune branche mais est éventuellement associée à une ou plusieurs traces d'électrons lents (électrons Auger) ou à un recul.

Sur 576 mésons observés, nous avons trouvé 375 mésons σ, soit (65.1 ± 3.4) %, et 201 mésons ρ, soit (34.9 ± 2.5) %. Nous n'avons observé aucun cas de désintégration π-μ. Pour permettre la comparaison de nos résultats avec ceux d'autres auteurs (voir Tableau I) nous avons aussi déterminé les pourcentages de mésons σ et ρ en adoptant une longueur maximum de 6 μm pour les traces considérées comme reculs (cf. *b*).

TABLEAU I. – *Pourcentages de mésons σ et ρ.*

Auteurs		*R* max des reculs (μm)	Nombre de mésons	Pourcentages (*)	
				σ	ρ
PERKINS (1)		(?)	120	71	29
ADELMAN et JONES (10) . . .		0	512	73	27
CHESTON et GOLDFARB (8)	(*a*)	6	429	65	35
	(*b*)	0	»	81	19
MENON *et al.* (2)		3	~ 500	72	28
DE SABBATA *et al.* (4) . . .		5	~ 300	73	27
BENEVENTANO *et al.* (9) . . .		6	~ 500	70	30
Nos résultats	(*a*)	2	576	65	35
	(*b*)	6	»	61	39

(*) Arrondis au pourcent.

On voit que le résultat est assez sensible au choix de cette longueur maximum et que nous avons trouvé un pourcentage de mésons ρ plus élevé que les autres auteurs.

3·3. *Classement des étoiles d'aprés le nombre de branches.* – 1038 étoiles (observées sur 159 mm²) ont été classées d'après le nombre de branches (Tableau II).

(10) F. L. ADELMAN et S. B. JONES: *Science*, **111**, 226 (1950).

TABLEAU II. – *Distribution des étoiles d'après le nombre de branche.*

Nombre de branches	1	2	3	4	5	6	7
Nombre d'étoiles	415	283	210	105	25	8 (*)	0 (*)
Pourcentage	40.0 ± 2.0	27.3 ± 1.7	20.2 ± 1.4	10.1 ± 1.0	2.4 ± 0.5	0.06 ± 0.02 (*)	0 ± 0.01 (*)

(*) Valeur déduite de l'exploration rapide d'une surface de 2 200 mm².

Le tableau III permet de comparer nos résultats à ceux publiés par d'autres auteurs. On voit que pour les étoiles à plus de 2 branches tous les résultats sont à peu près en accord. Pour les étoiles à 1 et à 2 branches il y a des désaccords importants avec certains auteurs. Il semble qu'on puisse les attribuer essentiellement aux différentes conventions adoptées pour la longueur maximum des traces considérées comme reculs.

TABLEAU III. – *Distribution des étoiles d'après le nombre de branches.*

Auteurs	R_{max} des reculs (μm)	Nombre total d'étoiles	Pourcentages (*) d'étoiles à n branches					
			$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
ERKINS (1)	(?)	112	43	27	20	10	0.9	0
DELMAN et JONES (10)	0	375	30	37	21	11	2.5	0
HESTON et GOLDFARB (8) (a)	6	278	36	26	25	11	2.9	0
HESTON et GOLDFARB (8) (b)	0	348	37	27	23	11	2.3	0
IENON *et al.* (2) . .	3	1 802	34	31	22	11	1.9	0.1
ADELMAN (11)	(?)	1 631	32	33	22	11	2.0	0.1
NOMOTO *et al* (3) . .	5	200	40	28	22	10	1.0	0
DE SABBATA *et al* (4)	5	500	44	26	20	8.4	1.6	0
BENEVENTANO *et al.* (9)	6	1 634	46	26	18	7.9	1.4	0
Nos résultats	2	1 038	40	27	20	10	2.4	0.06

(*) Arrondis.

Nous trouvons un nombre moyen de branches par étoile $\overline{N}_b = 2.1$.

3·4. *Etoiles produites dans les noyaux légers et dans les noyaux lourds.* – Nous avons ensuite cherché à classer les 1 038 étoiles observées en deux catégories, suivant qu'elles ont été produites dans un noyau léger (C, N, O) ou dans un noyau lourd (Br, Ag) de l'émulsion.

(11) F. L. ADELMAN: *Phys. Rev.*, 85, 249 (1952).

Dans ce but, nous avons repris, en les précisant, les critères déjà utilisés par d'autres auteurs [1-4].

3·4.1. Charge totale des étoiles. En ce qui concerne les étoiles produites dans les noyaux légers, il est vraisemblable que dans la plupart des cas toutes les particules chargées résultant de la désintégration reçoivent une énergie cinétique suffisante pour donner lieu à une trace visible, étant donné que l'énergie propre du méson absorbé est dans tous les cas supérieure à l'énergie de liaison du noyau et que l'énergie d'excitation se répartit entre un petit nombre de particules de masse petite. Le tableau suivant indique, pour différentes particules, que l'énergie suffisante pour donner lieu à une trace nettement visible ($\sim 2\ \mu$m) est très petite par rapport à l'énergie d'excitation

Particule	Énergie pour $R = 2\ \mu$m
p	~ 0.2 MeV
α	~ 0.6 »
Li	~ 1.5 »
C	~ 2.4 »

On ne peut toutefois pas exclure la possibilité que l'énergie d'excitation se répartisse entre une partie seulement des particules résultant de la désintégration, mais nous supposons que ce cas est rare.

Dans ces conditions, nous faisons l'hypothèse que la charge totale visible des étoiles produites dans le carbone, l'azote ou l'oxygène est respectivement égale à 5, 6 ou 7 e, une charge étant annihilée par la charge négative du méson absorbé.

En ce qui concerne les étoiles produites dans les noyaux lourds, l'énergie propre du méson absorbé étant dans tous les cas de beaucoup inférieure à l'énergie de liaison du noyau, la réaction n'affecte qu'une petite partie du noyau. Si l'on admet que dans un « processus primaire » le méson interagit avec une paire de nucléons ou avec une particule α, il en résulte seulement une seule particule de charge e, et 1, 2 ou 3 neutrons. Le noyau restant peut encore émettre un petit nombre de particules chargées, soit par chocs directs des nucléons résultant du processus primaire, soit par évaporation. Dans cette dernière hypothèse et en admettant que l'énergie d'excitation du noyau est de 140 MeV (ce qui est évidemment un maximum), on peut voir, d'après LE COUTEUR [12], que les nombres moyens de particules chargées émises sont les suivants:

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[12] K. J. LE COUTEUR: *Proc. Phys. Soc.*, A **63**, 259 (1950).

Particules	Nombre moyen par étoile
protons	1.75
deutons	0.15
tritons	0.15
particules α	0.70
total	2.75

La charge totale moyenne est par conséquent égale à 3.45 e seulement.

Ces chiffres doivent évidemment être pris avec réserve car deux circonstances limitent l'applicabilité de la théorie de l'évaporation: l'interaction forte du méson π^- avec un petit nombre de nucléons seulement, et l'énergie d'excitation peu élevée.

On peut en conclure que la charge totale des étoiles produites dans les noyaux lourds sera généralement inférieure à $5e$. Il y a donc là un premier critère de discrimination des étoiles.

3·4.2. Barrière de potentiel des noyaux lourds. Alors que les particules chargées émises dans les désintégrations des noyaux légers n'ont en principe aucune limitation inférieure d'énergie, celles qui sont émises dans les désintégrations des noyaux lourds ont à franchir la barrière de potentiel du noyau résiduel, ce qui impose des limites inférieures à leurs énergies d'émission. Il y a donc là un second critère de discrimination des étoiles. Ce critère a été utilisé précédemment par divers auteurs [1,2,4,11] avec les limites inférieures d'énergie suivantes: 4 MeV pour les protons et 9 MeV pour les particules α, ce qui

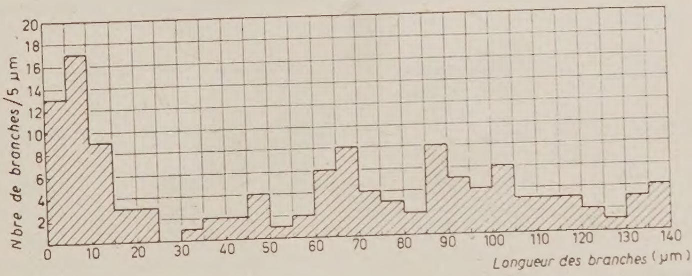


Fig. 1. – Distribution des longueurs des branches des étoiles à 1 branche.

correspond à des parcours respectifs de 120 µm et 50 µm environ. Nous avons cherché à préciser expérimentalement ces valeurs en examinant la distribution des longueurs des branches de toutes les étoiles à une seule branche. En effet, d'après le critère (a) ci-dessus, les particules provenant des noyaux légers doivent

TABLEAU IV.

Pourcentages d'étoiles produites dans les noyaux légers et dans les noyaux lourds

Auteurs	Nombre d'étoiles	Pourcentages légers	Pourcentages lourds
PERKINS (1)	112	53.5	46.5
MENON *et al.* (2)	~ 500	46.0	54.0
ENOMOTO *et al.* (3)	200	46.5	39.0
		indéterminés : 14.5	
DE SABBATA *et al.* (4)	500	46.2	53.8
Nos résultats	1 038	44.5	55.5

être des fragments de charge 5, 6 ou 7 e. On peut voir aisément que les énergies maxima de tels fragments correspondent à des parcours inférieurs à 26 μm, en raison de la conservation de l'énergie et de la quantité de mouvement. Ce parcours est vraisemblablement inférieur au parcours minimum des particules α pouvant être émises par les noyaux lourds, de sorte que l'on peut s'attendre à ce qu'il y ait moyen de distinguer les étoiles dues aux noyaux légers des étoiles dues aux noyaux lourds par la longueur de leur branche.

La Fig 1 montre que nous avons observé une telle séparation puisqu'il n'y a aucune branche de longueur comprise entre 25 μm et 30 μm.

On peut en conclure que les limites inférieures des parcours des particules pouvant être émises par les noyaux lourds doivent être abaissées à environ 30 μm pour les particules α et à environ 80, 55 et 45 μm respectivement, pour les protons, deutons et tritons. Ces parcours correspondent

TABLEAU V. – *Distribution d'aprés le nombre de branches des é*

Auteurs	Nombre d'étoiles	Pourcentage $n = 1$ légers	$n = 1$ lourds	$n = 2$ légers	$n = 2$ lourds
PERKINS (1)	112	3.6	39	22	5.4
MENON *et al.* (2)	~ 500	11	27	13	18
ENOMOTO *et al.* (3)	200	9.0	25	15	10
		indéterminés : 6.0		3.0	
DE SABBATA *et al* (4)	500	8.8	35	13	13
Nos résultats	1 038	4.3	36	11	16

(*) Arrondis.

à une énergie cinétique d'environ 6.5 MeV pour les particules α et d'environ 3.3 MeV pour les protons, deutons et tritons.

3'4.3. Electrons Auger. Des considérations théoriques montrent que les électrons Auger associés aux cascades mésiques sont plus nombreux dans les noyaux légers que dans les noyaux lourds. Mais leurs énergies étant plus basses, le seuil d'énergie (de 10 keV environ) qui sépare les électrons observables dans l'émulsion nucléaire de ceux qui ne le sont pas, réduit fortement la fréquence des électrons Auger visibles associées aux étoiles produites dans les noyaux légers. D'autre part, les électrons Auger de grande énergie, correspondant aux transitions à n petit, sont très rares dans les noyaux lourds car les transitions radiatives sont beaucoup plus probables (loi en Z^4). On peut donc s'attendre à observer des électrons Auger d'aspect semblable associés aux étoiles produites dans les noyaux lourds ou légers; seule la fréquence relative diffère. On peut estimer qu'il y a émission d'électrons Auger observables dans environ 50% des étoiles produites dans les noyaux lourds et dans environ 2% des étoiles produites dans les noyaux légers. Il n'y a donc pas là un critère de discrimination absolu mais seulement des probabilités très différentes.

* * *

Nous avons classé les 1038 étoiles observées en appliquant simultanément les critères 3'4.1, 3'4.2 et 3'4.3 ci-dessus.

Nous n'avons trouvé aucune étoile pour laquelle il y avait contradiction entre ces critères. Les étoiles pour lesquelles ces critères étaient insuffisants

s dans les noyaux légers et dans les noyaux lourds.

à n branches					
$n = 3$		$n = 4$		$n = 5$	
[illegible]gers	lourds	légers	lourds	légers	lourds
[illegible]8	1.8	8.9	0,9	0.9	0
[illegible]15	4.0	6.0	4.9	1.0	1.0
[illegible]15	3.0	7.0	1.0	0.5	0
3.5		1.5		0.5	
[illegible]16	3.2	6.4	2.0	1.4	0.2
[illegible]17	3.6	9.8	0.3	2.4	0

ont été classées dans les mêmes proportions que les étoiles des mêmes types ayant pu être classées sans ambiguïté; ce cas ne s'est présenté que pour un petit nombre d'étoiles à 3, 4 ou 5 branches.

Nous avons ainsi trouvé 462 étoiles produites dans les noyaux légers, soit $44.5 \pm 2.1\%$, et 576 étoiles produites dans les noyaux lourds, soit $55.5 \pm 2.3\%$.

Le Tableau IV montre que ces résultats globaux sont en accord avec ceux publiés par d'autres auteurs, y compris ceux d'Enomoto *et al.*, à condition d'admettre que les cas qu'ils ont laissés indéterminés peuvent être attribués à des noyaux lourds.

Le Tableau V donne les distributions d'après le nombre de branches des étoiles produites dans les noyaux légers et dans les noyaux lourds. On voit qu'il y a ici plusieurs désaccords importants, de sorte que l'accord constaté dans le Tableau IV apparaît comme fortuit. Ces désaccords semblent dus essentiellement au fait que les critères de discrimination appliqués par les différents auteurs ne sont pas identiques. Par exemple, si nous appliquons aux étoiles à une branche, le critère de la barrière de potentiel avec les conventions de MENON *et al.* (2), nous trouvons $7.2 \pm 0.8\%$ d'étoiles produites dans les noyaux légers et $33 \pm 2\%$ produites dans les noyaux lourds, valeurs qui sont plus proches de celles trouvées par ces auteurs.

4. – Désintégrations des noyaux légers.

4·1. *Distribution des étoiles.* – Le Tableau VI donne la distribution d'après le nombre de branches des 462 étoiles produites dans les noyaux légers.

TABLEAU VI.

Distribution d'après le nombre de branches des étoiles produites dans les noyaux légers.

Nombre de branches	1	2	3	4	5	6	7
Nombre d'étoiles	45	117	173	102	25	8 (*)	0 (*)
Pourcentage	9.7 ± 1.5	25.3 ± 2.3	37.4 ± 2.8	22.1 ± 2.5	3.4 ± 1.1	0.13 ± 0.05 (*)	0 ± 0.02 (*)

(*) Valeurs déduites de l'exploration rapide d'une surface de 2 200 mm².

4·2. *Modes de désintégration.* – En adoptant le classement des branches d'étoiles en trois catégories (cfr. Sect. 2·1): protons, particules α et fragments, on voit qu'il y a 22 modes de désintégration possibles, qui sont indiqués dans la 2ème colonne du Tableau VII. Les modes de désintégration réunis par

une accolade sont difficilement discernables entr'eux, à cause de la difficulté de discrimination entre les particules α et les fragments.

La 3ème colonne du Tableau VII donne les pourcentages observés de ces différents modes de désintégration, sans distinction entre les modes réunis par une accolade; la 4ème colonne donne une estimation des pourcentages partiels de ces derniers, basés sur une tentative de discrimination entre particules α et fragments.

L'existence d'étoiles à 2, 3, 4 et 5 branches dont l'une est un fragment

TABLEAU VII. – *Pourcentages des modes de désintégration des noyaux légers.*

Nombre de branches	Modes de désintégration	Pourcentages	
		sans discrimination α, F	avec discrimination α, F
1	F	9.7	9.7
2	Fp	15	15
	Fα	9.8	6.6
	2F		3.2
3	F2p	12	12
	2αp		14
	Fαp	23	8
	2Fp		1
	3α	2.5	1.9
	F2α		0.6
4	α3p	8.4	6.9
	F3p		1.5
	2α2p	9.7	4.5
	Fα2p		5.2
	3αp	3.8	3.8
5	5p	0.6	0.6
	α4p	1.5	1.1
	F4p		0.4
	2α3p	3.2	3.2
6	6p	0.03 (*)	0.03 (*)
	α5p	0.09 (*)	0.09 (*)
7	7p	0	0

(*) Valeurs déduites de l'exploration rapide d'une surface de 2 200 mm².

n'est établie de façon certaine que dans les cas où ce fragment est instable et donne lieu à un « marteau » ou à une désintégration β.

5 étoiles comportant un fragment donnant lieu à un « marteau » ont été trouvées parmi les 462 étoiles observées. 40 autres ont été trouvées au cours de l'exploration rapide d'une surface de 2 200 mm². Le Tableau VIII donne la distribution de ces 45 étoiles suivant leur mode de désintégration. Aucune désintégration β n'a été observée, mais nous ne sommes pas certains de voir toujours les traces d'électrons rapides, à cause du sous-développement des plaques.

TABLEAU VIII. – *Distribution des étoiles comportant un « marteau ».*

Nombre de branches	Modes de désintégration	Nombres d'étoiles
2	Fα	8
3	F2p	22
	Fαp	7
4	F3p	2
	Fα2p	5
5	F4p	1

Le schéma de Clark et Ruddlesden (5) appliqué aux noyaux de carbone, d'azote et d'oxygène conduit aux modes de désintégration indiqués dans la 2ème colonne du Tableau IX. Les 3ème et 4ème colonnes du tableau don-

TABLEAU IX. – *Pourcentages de désintégration suivant le schéma de Clark et Ruddlesden.*

Noyau	Modes de désintégration	Pourcentages	
		sans discrimination α, F	avec discrimination α, F
$^{12}_{6}C$	2α p	23	14
$^{14}_{7}N$	3α	2.5 } 12.2	1.9 } 6.4
	2α 2p	9.7	4.5
$^{16}_{8}O$	3α p	3.8	3.8
		Total: 39.0	Total: 24.2

nent les pourcentages observés, sans ou avec discrimination entre particules α et fragments, repris dans les colonnes correspondantes du Tableau VII. (Les pourcentages indiqués dans la 3ème colonne doivent être considérés commes des valeurs maxima).

On remarque tout d'abord que le schéma de Clark et Ruddlesden ne rend compte que d'un pourcentage total assez bas des événements (24.2 ÷ 39.0 %). Ensuite, le pourcentage très bas de désintégration de l'oxygène suivant le mode $3\alpha p$ est en contradiction avec ce qu'on pouvait attendre des abondances relatives du carbone, de l'azote et de l'oxygène dans l'émulsion nucléaire (4.66 / 1 / 2.55). De même, il est surprenant que la désintégration de l'azote suivant le mode 3α (résultant de l'interaction du méson π^- avec le deuton) est désavantagé par rapport au mode $2\alpha\,2p$. Ces deux faits suggèrent que le noyau intermédiaire de ${}^{12}_{6}C$ apparaissant dans les modes de désintégration $3\alpha p$ et 3α pourrait subsister, étant donné sa stabilité bien connue. Un certain pourcentage des désintégrations suivant les modes F et Fp pourraient correspondre à ces deux cas.

D'autre part, on observe un pourcentage assez élevé (11.8 ÷ 13.7 %) d'autres modes de désintégration ne donnant lieu qu'à des particules α et à des protons comme particules chargées (voir Tableau VII).

Nous sommes ainsi amenés à considérer une extension du schéma de Clark et Ruddlesden dans laquelle la structure en particules α est préservée mais où: 1) trois particules α peuvent rester liées pour donner un fragment ${}^{12}_{6}C$; 2) plus d'une particule α peut être brisée. Ce « modèle α » appliqué aux noyaux de carbone, d'azote et d'oxygène, conduit aux modes de désintégration indiqués dans le Tableau X. On voit qu'il rend compte d'environ les $\frac{2}{3}$ des désintégrations observées.

Le pourcentage de désintégrations attribuées à l'azote semble trop élevé, mais il n'est pas improbable que la plupart des désintégrations suivant les modes restants (Fα, 2F, F2p, ...) puissent être attribuées au carbone et à l'oxygène. En effet, les valeurs possibles de la charge des fragments émis dans ces modes de désintégration sont 3, 4 et 5 e, correspondant respectivement à un noyau initial de carbone, d'azote et d'oxygène. Or, le lithium et le bore possèdent chacun deux isotopes stables (${}^{6}_{3}L$ ${}^{7}_{3}L$ et ${}^{10}_{5}B$ ${}^{11}_{5}B$) alors que le béryllium n'en possède qu'un (${}^{9}_{4}Be$) qui contient un neutron faiblement lié.

4·3. *Spectres d'énergie des particules* α. – CLARK et RUDDLESDEN (5) ont traité la désintégration des noyaux du type ${}^{4k}_{2k}X$ suivant le schéma:

$$(1)\qquad {}^{4k}_{2k}X + \pi^- \to (k-1)\alpha + \begin{cases} p + 3n & (a), \\ d + 2n & (b), \\ t + n & (c), \end{cases}$$

($k = 3$ correspond à ${}^{12}_{6}C$ et $k = 4$ à ${}^{16}_{8}O$).

TABLEAU X. – *Pourcentages de désintégrations suivant le «modèle α».*

Noyau	Modes de désintégration	Pourcentages sans discrimination α, F		Pourcentages avec discrimination α, F	
$^{12}_{6}C$	2α p	23	32.0	14	21.5
	α 3p	8.4		6.9	
	α 5p	0.6		0.6	
$^{14}_{7}N$	($^{12}_{6}C$)	9.7	22.0	9.7	15.8
	3α	2.5		1.9	
	2α 2p	9.7		4.5	
	α 4p	1.5		1.1	
	6p	0.03		0.03	
$^{16}_{8}O$	($^{12}C_6$)p	15	22.1	15	22.1
	3α p	3.8		3.8	
	2α 3p	3.2		3.2	
	α 5p	0.09		0.09	
	7p	0		0	
		Total: 77.5		Total: 60.8	

Ils ont en plus examiné le cas particulier

$$^{12}_{6}C + \pi^- \to {}^{8}_{4}Be + \{\ \},$$

dans le but d'étudier l'influence du $^8Be_4$ intermédiaire sur la corrélation angulaire entre les deux particules α.

La fonction d'onde des noyaux $^{4k}_{2k}X$ est décomposée en un produit de $(k+3)$ fonctions d'onde: $(k-1)$ fonctions d'onde de particules α décrites globalement (coordonnées de leur centre de masse) et les 4 functions d'onde des nucléons de la k-ième particule α. (C'est un des deux protons de celle-ci qui est transformé en neutron en interagissant avec le π^-).

Le spectre d'énergie cinétique des particules α, obtenu par Clark et Ruddlesden, s'écrit:

$$(2) \qquad P(E)\,dE \doteq e^{-\gamma t}(1-t)^m t^{\frac{1}{2}}\,dt \int_0^{\gamma(1-t)} e^{-u}\left[1-\frac{u}{\gamma(1-t)}\right]^m u^{((3k/2)-4)}\,du,$$

où $t = E/E_{max}$ (varie de 0 à 1); γ est un paramètre dépendant de la liaison des k particules α; $m = \frac{7}{2}$, 2 ou $\frac{1}{2}$ suivant qu'il s'agit de la réaction (1,a), (1,b)

ou (1,c). Cette méthode s'étend aisément au cas des noyaux ${}^{4k+2}_{2k+1}X$ en les supposant constitués de k particules α et d'un deuton. On peut alors avoir les trois types de réaction:

$$(3) \qquad {}^{4k+2}_{2k+1}X + \pi^- \to \begin{cases} (k-1)\alpha + d + \{\,\}\,, & (a)\,, \\ (k-1)\alpha + p + n + \{\,\}\,, & (b)\,, \\ k\alpha + 2n\,, & (c)\,, \end{cases}$$

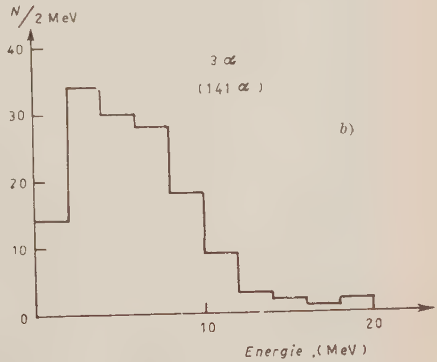
($k = 3$ correspond à ${}^{14}_{7}N$).

Dans le cas de la réaction (3,c), l'exposant ($\frac{3}{2}k - 4$) de l'expression (2) doit être remplacé par ($\frac{3}{2}k - \frac{5}{2}$). Quant à l'exposant m, il vaut:

$$5,\ \tfrac{7}{2} \text{ ou } 2 \text{ dans le cas } (3,a)\,,$$

$$\tfrac{13}{2},\ 5 \text{ ou } \tfrac{7}{2} \text{ dans le cas } (3,b)\,,$$

$$\text{et } \tfrac{1}{2} \text{ dans le cas } (3,c)\,.$$



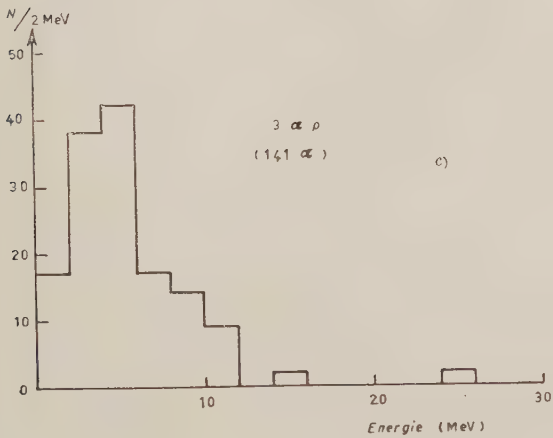


Fig. 2. - a) Spectre d'énergie des particules α de la désintégration 2αp. La courbe tracée représente le spectre « standard » de Clark et Ruddlesden ($\gamma = 12$). b) Spectre d'énergie des particules α de la désintégration 3α. c) Spectre d'énergie des particules α de la désentegration 3 α p.

On se rend compte aisément de ce que tous ces spectres diffèrent peu les uns des autres et que, dans les conditions expérimentales présentes, on ne peut espérer mettre en évidence des différences entre les spectres correspondant aux différentes réactions possibles. Cette difficulté apparait encore mieux si l'on songe au fait que les différentes valeurs du paramètre γ ne sont pas connues avec précision.

Nous avons établi expérimentalement les spectres d'énergie des particules α dans les étoiles des types 2αp, 3α et 3αp (Figs. 2, *a*, *b* et *c*). Les énergies ont été déterminées par la mesure du parcours des particules α. On voit que l'allure de ces spectres confirme qualitativement les prévisions théoriques. Ils sont en bon accord avec ceux obtenus précédemment par MENON *et al.* ([2]) et par LEDERMAN *et al.* (communication privée).

5. – Conclusions.

Les étoiles produites dans les noyaux lourds et légers de l'émulsion nucléaire ont pu être séparées sans difficulté, en appliquant trois critères cohérents.

L'identification des noyaux légers primaires a présenté des difficultés plus grandes, qui n'ont pu toujours être résolues, en particulier lorsque l'étoile comporte une ou plusieurs traces courtes.

Les deux tiers environ des désintégrations produites dans les noyaux légers s'interprètent aisément en supposant que ces noyaux conservent une structure en particules α, même lorsque l'énergie d'excitation est assez élevée. Les autres désintégrations donnent lieu à des fragments nucléaires et proviennent vraisemblablement de la désintégration de noyaux ${}_5B^*$ et ${}_7N^*$ excités. Dans ce cas, le méson interagit avec un proton sans que l'énergie apportée se répartisse également entre les autres nucléons. Un tel schéma d'échauffement local a été proposé antérieurement pour les noyaux légers par PERKINS ([13]) et pour les noyaux lourds par HEIDMANN et LEPRINCE-RINGUET ([14]) et par PUPPI *et al.* ([4]).

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Nous remercions vivement les Professeurs P. BAUDOUX et J. GEHENIAU

([13]) D. H. PERKINS: *Nature*, **161**, 487 (1948).

([14]) J. HEIDMANN et L. LEPRINCE-RINGUET: *Compt. Rend. Ac. Sc. Paris*, **226**, 1716 (1948).

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RIASSUNTO (*)

Lastre Ilford G.5 sono state esposte ai mesoni π^- lenti prodotti dal ciclotrone di Rochester. Circa 1 500 stelle σ sono state classificate secondo il numero dei rami e secondo che sono state prodotte in un nucleo leggero (C, N, O) o in un nucleo pesante (Br, Ag) dell'emulsione. L'esame delle stelle prodotte nei nuclei leggeri ha permesso di studiare i modi di disintegrazione di tali nuclei. Si propone un'estensione dello schema di Clark e Ruddlesden che dà ragione di circa $\frac{2}{3}$ delle stelle osservate. Le restanti stelle contengono almeno un frammento ($Z > 2$) e sembra si possa interpretarle meglio con un meccanismo di riscaldamento locale.

(*) *Traduzione a cura della Redazione.*

About the Possible Annihilation Mode of a Nucleon-Antinucleon System into a K-$\overline{\text{K}}$ Pair.

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(ricevuto il 23 Aprile 1956)

Summary. – The selection rules for annihilation of a nucleon-antinucleon system into a K-antiK pair which are due to conservation of angular momentum, parity (P), particle-antiparticle conjugation quantum number (C), and G quantum number conservation are discussed. It is shown that, in the framework of the Lee-Yang suggestion of a parity conjugation invariance, two new quantum numbers (called E and F) can be constructed, which give rise to additional selection rules.

1. – From the assumption that K^0 and K^+ form isotopic spin doublets it follows that their isotopic spin is $I_K = \frac{1}{2}$ and that their strangeness [1] is $S = 1$. Their antiparticles, $\overline{K}^0$ and K^-, must have strangeness $S = -1$. Protonium (the possible bound system formed by a proton and an antiproton, the analogue of positronium [2]) can annihilate into a K-$\overline{K}$ pair without violating the strangeness selection rule.

The possible reactions are

$$\bar{p} + p \rightarrow K^- + K^+ , \tag{1'}$$

$$\bar{p} + p \rightarrow \overline{K}^0 + K^0 , \tag{1''}$$

A possible strong (N, Λ^0, $K^{\overset{+}{0}}$) interaction (which could also be responsible for associated production of Λ^0 plus K^+ from ordinary particles, production

[1] M. Gell-Mann: *Proc. of the Pisa Conference* (1955).

[2] See: R. E. Marshak: *Meson Physics* (New York, 1952); L. M. Brown and M. Peshkin: *Phys. Rev.* (in the press) conclude that an appreciable fraction of bevatron antiprotons (~ 500 MeV) can be stopped in hydrogen and form protonium.

of Λ^0 following K^- absorption by nucleons, a K-N potential, binding of Λ^0 to nucleons) and a possible strong (N, Σ, $K^{\overset{+}{0}}$) interaction (which could also be responsible for associated production of Σ plus $K^{\overset{+}{0}}$, production of Σ from K^- absorption, a K-N potential, a shorter range Σ-N potential) may produce (1′) and (1″)—consider, for instance, the virtual sequences: $\bar{p} + p \to \bar{p} + \Sigma$ (or Λ^0) $+ K^+ \to K^- + K^+$, $\bar{p} + p \to \bar{p} + \Sigma^+ + K^0 \to \bar{K}^0 + K^0$. In general the modes of annihilation (1) will have to compete with the modes of annihilation into 2, 3, 4, 5 pions which may have comparatively large rates. However, a number of selection rules are effective for annihilation from the protonium bound states—these rules forbid for instance two pion annihilation from certain states, whereas two K annihilation may be permitted.

There is a groving evidence (3) that the $K^+_{3\pi}$ events and the $K^+_{2\pi}$ events are not due to the same particle—this evidence is obtained on the basis of DALITZ's (4) discussion of the $K^+_{3\pi}$ decay. Although the spin-parity assignment to the τ^+ meson (responsible for the $K^+_{3\pi}$ events) is still not definite, the 0^- assignment seems the most consistent choice among the lowest spin values (5). LEE and YANG (6) have suggested a symmetry principle implying that θ (θ^+ responsible for the $K^+_{2\pi}$ events) and τ have nearly equal masses, same spin and opposite parity. In the following we limit the discussion to τ-meson pseudoscalar, θ-meson scalar.

2. – Selection Rules for Annihilation of a p-$\bar{p}$ System into a K-$\bar{K}$ Pair. Annihilation from the Protonium Bound States.

We discuss the selection rules due to conservation of total angular momentum, total parity, and charge conjugation quantum number (7). We classify the initial $\bar{p}p$ states into the four groups: singlets, triplets with $J = L - 1$, triplets with $J = L$, and triplets with $J = L + 1$ (J = total angular momentum, L = relative orbital angular momentum for the $\bar{p}p$ system). The final $\bar{K}$-K pair will belong to one of the two following groups:

first group: $\tau + \bar{\tau}$, $\theta + \bar{\theta}$ (i.e.: $\tau^+ + \tau^-$, $\theta^+ + \theta^-$, $\tau^0 + \bar{\tau}^0$, $\theta^0 + \bar{\tau}^0$),

second group: $\tau + \bar{\theta}$, $\theta + \bar{\tau}$ (i.e.: $\tau^+ + \theta^-$, $\theta^+ + \tau^-$, $\tau^0 + \bar{\theta}^0$, $\theta^0 + \bar{\tau}^0$).

(3) See: B. BHOWMIK, D. EVANS, I. J. VAN HEERDEN and D. J. PROWSE: *Nuovo Cimento*, **3**, 574 (1956); similar results have been obtained at Berkeley, at Columbia, and at MIT.

(4) R. H. DALITZ: *Phil. Mag.*, **44**, 1068 (1953); *Phys. Rev.*, **94**, 1046 (1954).

(5) See ref. (3). The 2^- assignment cannot be rejected on such arguments—however a 2^- τ-meson might have a dominant mode of decay into $\pi^+ + \gamma$, not yet observed.

(6) T. D. LEE and C. N. YANG: *Phys. Rev.* (in the press).

(7) A. PAIS and R. JOST: *Phys. Rev.*, **87**, 871 (1952); L. WOLFENSTEIN and D. G. RAVENHALE: *Phys. Rev.*, **88**, 279 (1952); L. MICHEL: *Nuovo Cimento*, **10**, 319 (1953).

In Table I (P) means: forbidden by parity conservation; (C) means: forbidden by charge conjugation conservation.

TABLE I. – *Selection rules for annihilation of* $\bar{\text{p}}+\text{p}$ *into* $\bar{\text{K}}+\text{K}$.

Initial stae of the $\bar{\text{p}}$-p system		Final state $\tau+\bar{\tau}$ or $\theta+\bar{\theta}$	Final state
singlets		(P)	
triplets	$J=L-1$		(P)
	$J=L$	(P) (C)	(E)
	$J=L+1$		(P)

Singlet $\bar{\text{p}}$-p states and triplets $\bar{\text{p}}$-p states with $J=L$ cannot decay into two pions [8]. These states are not forbidden to decay into $\tau+\bar{\theta}$ or $\theta+\bar{\tau}$. It will be noted that, for a singlet-triplet single gamma transition the initial and final L must be both even or both odd—as required by charge conjugation invariance—and therefore the transition is forbidden for electric dipole. On the other hand, singlet-singlet single gamma transitions are not forbidden for electric dipole if $\Delta L = \pm 1$.

If two pion emission plays an important role in the annihilation of protonium, its absence in the case of annihilation from singlet states and from triplet states with $J=L$, may alter the competition in the direction of favoring the $\bar{\text{K}}+\text{K}$ annihilation modes—any quantitative estimation would however be very uncertain at present. A peculiar aspect of the annihilation of a $\bar{\text{p}}$-p pair into a $\bar{\text{K}}$-K pair is that, even if it occurs in a nucleus, a K^+ or a K^0 (free, or in same cases—if K-fragments exist—bound) and a K^-, or a $\bar{\text{K}}^0$, or a Σ, or a Λ must always emerge from the star—a consequence a strangeness conservation.

3. – Selection Rules for Annihilation of a $\bar{\text{p}}$-n System into a $\bar{\text{K}}$-K Pair.

The initial state is not an eigenstate of the charge conjugation operator. However it is an eigenstate of the operator $G = C\exp[i\pi I_2]$ (C = charge

[8] L. MICHEL: op. cit.; D. AMATI and B. VITALE: *Nuovo Cimento*, **2**, 719 (1955); T. D. LEE and C. N. YANG: *Nuovo Cimento*, **3**, 349 (1956).

conjugation, I_2 = 2-component of the isotopic spin operator) [9]. The final states are:

first group: $\tau^- + \tau^0$, $\theta^- + \theta^0$,

second group: $\tau^- + \theta^0$, $\theta^- + \tau^0$.

In Table II (P) means: forbidden by parity conservation; (G) means: forbidden by G conservation.

TABLE II. – *Selection rules for annihilation of* $\overline{\text{p}}+\text{n}$ *into* $\overline{\text{K}}$-K.

Initial states of the $\overline{\text{p}}$-n system		Final state	Final state
singlets		(P)	
triplets	$J = L - 1$		(P)
	$J = L$	(P) (G)	(F)
	$J = L + 1$		(P)

4. – The Suggested Conservation of Parity Conjugation Quantum Number. The Quantum Numbers E and F.

LEE and YANG [6] suggest that the part of the hamiltonian which includes the strong interactions commutes with a parity conjugation operator C_P

$$[C_P, H_{\text{strong}}] = 0 .$$

Moreover C_P commutes or anticommutes with P according to the strangeness S being even or odd respectively. We define two operators

$$E = CC_P$$

and

$$F = GC_P = C \exp[i\pi I_2] C_P .$$

([9]) See A. PAIS and R. JOST: loc. cit.; L. MICHEL: op. cit., L. RADICATI: unpublished: T. D. LEE and C. N. YANG: loc. cit.

Let us first consider $\bar{p}$-p annihilation. For this case we have E as a new quantum number, and we obtain the additional selection rule that $\bar{p}$-p triplets with $J = L$ cannot annihilate into $\tau + \bar{\theta}$ or $\theta + \bar{\tau}$ (see Table I). In the case of a $\bar{p}$-n system F is a new quantum number, which provides the additional selection rule that $\bar{p}$-p triplets with $J = L$ cannot annihilate into $\tau^- + \theta^0$ or $\theta^- + \tau^0$.

Parity conjugation conservation would provide new selection rules in the case of systems of even strangeness. For example, possible bound $\Lambda K^{\overset{+}{0}}$ systems with $C_P = -1$ $(\Lambda_1\tau - \Lambda_2\theta,\ \Lambda_1\theta - \Lambda_2\tau)$, possible bound $\overline{K}K$ systems with $C_P = -1$ $(\bar{\tau}\tau - \bar{\theta}\theta,\ \bar{\tau}\theta - \bar{\theta}\tau)$, etc., could not disintegrate by fast interactions, and, if the γ interactions satisfy parity conjugation conservation, they would appear as metastable fragments. However, their production thresholds are rather high (for instance $N + N \to \Lambda + K + \Lambda + K$).

As a final remark we would like to point out that no experimental arguments exist so far in favor of the particular assignment of quantum numbers considered here. In a recent paper FOLDY [10] discusses the possibility of several distinct relativistic theories for particles of any given spin. Of course, the selection rules obtained in the present paper would be altered for a different assignment of quantum numbers.

* * *

The author wishes to thank Prof. N. KROLL for many useful discussions.

([10]) L. L. FOLDY: *Phys. Rev.* **102**, 568 (1956).

RIASSUNTO

Vengono studiate le regole di selezione che risultano dalla conservazione del momento angolare, della parità (P), del numero di coniugazione di carica (C), e del numero G nell'annichilamento di un sistema nucleone-antinucleone in una coppia di K-antiK. Si mostra come, nell'ambito della ipotesi di LEE e YANG di una invarianza per coniugazione di parità, si possono costruire due nuovi numeri quantici (che vengono chiamati E ed F), che dànno luogo a regole di selezioni addizionali.

Theory of Multiple Boson Production (*).

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(ricevuto il 21 Maggio 1956)

Summary. - A general formula for the multiple boson production is derived. The introduction of a statistical idea to the general formula will be made. Neither comparison to experimental data nor interpretation of them will be given, but the discussion is restricted to the treatment of the several theories proposed so far from a unifying view point, particularly the field theoretical foundation of Fermi's statistical theory will be discussed in detail.

1. – Introduction.

Fermi and Lewis-Oppenheimer-Wouthuysen (LOW) have studied the multiple meson production from different points of view (1) (+).

Fermi has assumed that the whole energy of two colliding nucleons is thrown into a volume Ω equal to the Lorentz contracted volume of pion field surrounding a nucleon, and then distributed to all the possible final states as if thermodynamics equilibrium existed in the Ω.

LOW, on the other hand, applied the method by Bloch and Nordsieck in analogy to the case of electromagnetic radiation. The total energy which will go into meson production is distributed among the different mesons in accordance with a calculated spectrum.

(*) This work is supported by the National Science Foundation.

(1) E. Fermi: *Prog. Theor. Phys.*, **5**, 570 (1951); H. W. Lewis, J. R. Oppenhemier and S. A. Wouthuysen: *Phys. Rev.*, **73**, 127 (1948); H. W. Lewis: *Rev. Mod. Phys.*, **24**, 241 (1952).

(+) The abbreviation « m.b.p. » will be used to call the « multiple boson production », since our discussion in this paper is not limited to the case of mesons but holds true for the production of any kind of bosons.

We have, in the previous paper ([2]), generalized LOW's idea and discussed problems of mesonic proper-field by introducing a quantity « meson spectrum » which plays an important role in this kind of problem.

The question, however, as to what relation exists between Fermi's and LOW's theories, which seem at a glance to be based on entirely different ideas, has been asked.

This paper will be devoted to show the relationship between them. In Sect. 2·1, the general formula for m.b.p. will be derived independently of any approximation, starting with the field equation for bosons which satisfy the causality condition. The formula is written in terms of Heisenberg operators only.

In Sect. 2·2, several lemmas on low energy bosons will be concluded from the formula.

We shall, in Sect. 3, introduce the statistical idea and determine the most probable probability of the m.b.p. by the variation with respect to the occupation number of bosons ([3]). It will be seen that the above procedure to obtain the most probable probability will be carried out in an analogous manner to the determination of the temperature in statistical mechanics. One of the reasons for writing the theory in this form is of course to show the field theoretical basis of Fermi's thermodynamic approach (*).

In Sect. 4, the comparison will be made of the general theory developed in Sect. 2 to theories appeared so far, especially Fermi's and LOW's theories ([4]).

We shall not discuss the experimental information, nor its interpretation. We shall also not consider, in this paper, some of the interesting problems to be examined: for instance, the angular distribution of produced bosons, the damping effect on the multiplicity of bosons, the role of the nuclear potential, the statistical correlation of bosons, etc.

2. – General Formula for the m.b.p.

2·1. *General expression of the S-matrix for the production of n bosons* (+). – We will start with the equation of motion for the boson field with the mass μ

$$(\Box - \mu^2)\boldsymbol{U}(x) = \boldsymbol{O}(x) \quad (\times) \tag{2.1}$$

([2]) H. UMEZAWA, Y. TAKAHASHI and S. KAMEFUCHI: *Phys. Rev.*, **85**, 505 (1952).

([3]) Y. TAKAHASHI: *Can. Journ. Phys.*, **34**, 378 (1956).

(*) The importance of this problem has been specially emphasized by Professor TAKETANI a few years ago.

([4]) D. ITO: *Prog. Theoret. Phys.*, in press.

(+) The essential part of the discussion in this section is presented in reference ([2]).

(×) $\boldsymbol{U}(x)$ may be any kind of bosons. In order to avoid the trouble of suffixes, we will work with the scalar field throughout. Our discussion will easily be generalized, for instance, to the case of multiple photon production.

and the equation of motion for the source field, which will not explicitly be written down. The source function $\boldsymbol{O}(x)$ may be any kind of fields, provided that it satisfies the condition

$$[\boldsymbol{O}(x), \boldsymbol{O}(y)] = 0 \qquad \text{for} \quad (x-y)^2 > 0. \tag{2.2}$$

$\boldsymbol{O}(x)$ will not be specified until it becomes necessary to do so.

A formal solution of (2.1) is immediately obtained by

$$\boldsymbol{U}(x) = U(x) + \int_{-\infty}^{\sigma} \Delta(x-x')\boldsymbol{O}(x')(\mathrm{d}x'), \qquad (x \in \sigma), \tag{2.3}$$

from which a new field is defined as follows,

$$U(x, \tau) = U(x) + \int_{-\infty}^{\tau} \Delta(x-x')\boldsymbol{O}(x')(\mathrm{d}x'). \qquad (x \notin \tau)\ (^{+}). \tag{2.4}$$

Since $U(x, \tau)$ satisfies, as is verified by the definition, the free equation of motion

$$(\Box - \mu^2)U(x, \tau) = 0, \qquad (x \notin \tau) \tag{2.5}$$

$U(x, \tau)$ can be split into two parts, i.e., the positive and the negative frequency parts,

$$U^{(\pm)}(x, \tau) = U^{(\pm)}(x) + \int_{\infty}^{\tau} \Delta^{(\pm)}(x-x')\boldsymbol{O}(x')(\mathrm{d}x'), \tag{2.6}$$

which play the role of annihilation and creation operators, respectively.

$U^{(+)}(x, \tau)$ and $U^{(+)}(x)$, on the other hand, are connected by a unitary transformation satisfying

$$i\frac{\delta S(\tau)}{\delta \tau(x)} = -\mathcal{L}_{\text{interaction}}(x)S(\tau), \tag{2.7}$$

namely,

$$U^{(+)}(x, \tau) = S^{-1}(\tau)U^{(+)}(x)S(\tau) ; \quad (*) \tag{2.8}$$

(+) If $x \in \tau$, $U(x, \tau) = \boldsymbol{U}(x)$. The symbol x/τ will be used if $x \in \tau$.

(*) Regarding the rigorous formulation of quantization, see Y. Takahashi and H. Umezawa: *Prog. Theor. Phys.*, **9**, 14, 501 (1954).

We have also

$$(2.9)\qquad [U(x,\tau),\,U(y,\tau)] = [U(x),\,U(y)] = i\Delta(x-y)\,.$$

Since we are interested in a process of the production of n bosons, we shall derive a general expression of the matrix element of the S-matrix for it.

The matrix element between states for which there is no boson initially and n bosons finally is, omitting the normalization factor

$$(2.10)\qquad S_n(x) = \left\langle \frac{1}{n!}\left\{U^{(+)*}(x,\infty)\right\}^n \Phi_0',\,\Phi_0\right\rangle,\ (^+)$$

where Φ_0 (or Φ_0') is the vacuum state for bosons and proper state for nucleons; for instance, Φ_0 is a state for nucleon pair and Φ_0' is a vacuum state, if we are interested in the production of n bosons due to a pair annihilation. We shall not specify Φ_0 and Φ_0'. If the bosons are produced by the potential scattering of the nucleon, the effect is also included in $\boldsymbol{O}(x)$.

In order to treat (2.10) *without recourse to perturbation theory*, let us first consider the following operator,

$$(2.11)\qquad \chi_n(x,\tau) = \frac{1}{n!}\left\{U^{(+)}(x,\tau)\right\}^n .$$

In the case where $\boldsymbol{O}$ contains no U-operator (*), $\chi_n(x,\tau)$ satisfies the equation

$$(2.12)\qquad i\,\frac{\delta\chi_n(x,\tau)}{\delta\tau(x')} = u(x;x')\chi_{n-1}(x,\tau)\,,$$

where

$$(2.13)\qquad u(x;x') \equiv i\Delta^{(+)}(x-x')\boldsymbol{O}(x')\,.$$

The solution of (2.12) is readily written down as

$$(2.14)\qquad S_n(x) = \langle\Phi_0',\,\chi_n(x,\infty)\Phi_0\rangle = (-i)^n\int_{-\infty}^{\infty}(\mathrm{d}x_1)\int_{-\infty}^{\tau_1}(\mathrm{d}x_2)\,\ldots\int_{-\infty}^{\tau_{n-1}}(\mathrm{d}x_n)\cdot$$

$$\cdot\langle\Phi_0',\,u(x;x_1)\,\ldots\,u(x;x_n)\Phi_0\rangle = \frac{(-i)^n}{n!}\int_{-\infty}^{\infty}\!\!\ldots\int(\mathrm{d}x_1)\,\ldots\,(\mathrm{d}x_n)\cdot$$

$$\cdot\langle\Phi_0',\,T[u(x;x_1),\,\ldots,\,u(x;x_n)]\Phi_0\rangle\,.$$

T is the well-known time-ordering operator defined by WICK (5) (×),.

(+) * stands for the Hermitian conjugation.

(*) See Appendix I.

(5) G. C. WICK: *Phys. Rev.*, **80**, 268 (1950).

(×) It is easily seen that (2.14) is equivalent to the corresponding matrix element of S-matrix introduced by LEHMANN *et al.* and COESTER. H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **1**, 205 (1955); F. COESTER: unpublished result.

It is interesting to notice that the equation (2.14) is written only in terms of operators in Heisenberg picture. This fact shows that the boson production takes place by the n-iteration of the source function $\boldsymbol{O}(x)$ and the mutual disturbance of one another. Note that the equation (2.14) is a generalized expression of the equation of the meson scattering obtained by LOW, NAMBU and others [6].

The equation (2.14) can be further discussed independently of any approximation.

First, let us note the identity

$$\Delta^{(+)}(x) = \partial_0\partial_0\int_{-\infty}^{\infty}\Delta^{(+)}(x-x')\eta(x')(\mathrm{d}x')\,, \tag{2.15}$$

where

$$\eta(x) \equiv \delta(t)\,\eta(\boldsymbol{x})\,, \qquad (\Delta-\mu^2)\,\eta(\boldsymbol{x}) = \delta(\boldsymbol{x})\,. \tag{2.16}$$

From (2.15) (2.14) and (2.13), we obtain

$$\begin{aligned}
S_n(x) = {} & (n\,!)^{-1}\int_{-\infty}^{\infty}\!\!\cdots\int(\mathrm{d}x_1)\ldots(\mathrm{d}x_n)(\mathrm{d}x_1')\ldots(\mathrm{d}x_n')\cdot\\
&\cdot\ddot{\Delta}^{(+)}(x-x_1')\ldots\ddot{\Delta}^{(+)}(x-x_n')\,\eta(x_1'-x_1)\ldots\eta(x_n'-x_n)\cdot\\
&\cdot\langle\Phi_0',\,T[\boldsymbol{O}(x_1)\ldots\boldsymbol{O}(x_n)]\Phi_0\rangle=\\
= {} & (n\,!)^{-1}\int_{-\infty}^{\infty}\!\!\cdots\int(\mathrm{d}x_1)\ldots(\mathrm{d}x_n)(\mathrm{d}x_1')\ldots(\mathrm{d}x_n')\cdot\\
&\cdot\dot{\Delta}^{(+)}(x-x_1')\ldots\dot{\Delta}^{(+)}(x-x_n')\,\eta(x_1'-x_1)\ldots\eta(x_n'-x_n)\cdot\\
&\cdot\langle\Phi_0',\,T^*[\dot{\boldsymbol{O}}(x_1)\ldots\dot{\boldsymbol{O}}(x_n)]\Phi_0\rangle\,,
\end{aligned} \tag{2.17}$$

where $\cdot$ stands for time derivative and T^* signifies that the time differentiation has to be carried out after the time ordering, i.e.,

$$T^*(\dot{A}(x)\dot{B}(x')) = \frac{\partial}{\partial t}\frac{\partial}{\partial t'}\,T(A(x)B(x'))\,. \tag{2.18}$$

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[6] F. LOW: *Phys. Rev.*, **97**, 1392 (1955); Y. NAMBU: *Phys. Rev.*, **98**, 803 (1955).

Introducing the Fourier transfrom of $\Delta^{(+)}$ and η

(2.18)
$$\begin{cases} \Delta^{(+)}(x) = \dfrac{-i}{(2\pi)^3}\displaystyle\int \dfrac{\mathrm{d}\boldsymbol{k}}{2K} \exp[+i\mathfrak{k}x]\,, \\ \eta(x) \quad = \dfrac{-1}{(2\pi)^4}\displaystyle\int \dfrac{(\mathrm{d}k)}{K^2} \exp[+ikx]\,, \end{cases}$$

where

(2.20)
$$\begin{cases} K \equiv \sqrt{\boldsymbol{k}^2 + \mu^2}\,, \\ \mathfrak{k} \; = (k_1, k_2, k_3, iK)\,, \end{cases}$$

we have

(2.21)
$$S_n(x) = (n\,!)^{-1}\int \dots \int \prod_{i=1}^{n} \left\{ \frac{\mathrm{d}\boldsymbol{k}_i}{(2\pi)^3} \frac{\exp[i\mathfrak{k}_i x]}{2K_i^2} \right\} \varphi(\mathfrak{k}_1, \dots \mathfrak{k}_n)\,,$$

where

(2.22)
$$\varphi(\mathfrak{k}_1, \dots \mathfrak{k}_n) \quad \int_{-\infty}^{\infty} \dots \int \prod_{i=1}^{n} \{(\mathrm{d}x_i) \exp[-i\mathfrak{k}_i x_i]\} \langle \Phi_0' T^*[\dot{O}(x_1) \dots \dot{O}(x_n)]\Phi_0 \;.$$

The probability for the production of n bosons with momenta $\boldsymbol{k}_1, \dots, \boldsymbol{k}_n$ is, therefore,

(2.23)
$$\mathrm{d}w_n = \prod_{i-1}^{n} \frac{\mathrm{d}\boldsymbol{k}_i}{(2\pi)^3 2K_i^3} \cdot |\varphi_p(\mathfrak{k}_1, \dots, \mathfrak{k}_n)|^2\,,$$

and the total probability turns out to be

(2.24)
$$w_n = (n\,!)^{-1} \int_{(\text{con.})} \dots \int \prod_{i=1}^{n} \frac{\mathrm{d}\boldsymbol{k}_i}{(2\pi)^3 2K_i^3} \cdot |\varphi_p(\mathfrak{k}_1, \dots, \mathfrak{k}_n)|^2\,,$$

where (con.) signifies that the integration is to be performed in such a way as to satisfy the momentum conservation law, and

(2.25)
$$\varphi_p(\mathfrak{k}_1, \dots, \mathfrak{k}_n) \equiv \sum_{(\text{permutation})} \varphi(\mathfrak{k}_1, \dots, \mathfrak{k}_n).$$

We are now ready to derive some lemmas for the low energy boson production.

2·2. *Some Consequence of the General Formula.* – In the preceding subaection we have derived general formulas (2.21) and (2.24) for the production sf n bosons in such a way as to be independent of any approximation as long os we start from the equation (2.1).

Let us discuss, before going into details, some consequences to be derived immediately from (2.21).

It is noted that $\varphi(\mathfrak{k}_1, ..., \mathfrak{k}_n)$ of (2.21) is expressed by the Fourier transform of time derivatives of a symmetric function $\langle \Phi_0', T(\boldsymbol{O}(x_1) ..., \boldsymbol{O}(x_n))\Phi_0\rangle$. Therefore, if one of the bosons, say $\mathfrak{k}_1$, is very soft, in other words, the frequency of a produced boson is small compared with the linear dimension of the time change of the source, the exponential function $\exp[-i\mathfrak{k}_1 x_1)$ can be put equal to 1 (*), then after the time integration we are left with

$$(2.26) \qquad \varphi(\mathfrak{k}_1, ..., \mathfrak{k}_n) = \int_{-\infty}^{\infty} ... \int (\mathrm{d}x_2) ... (\mathrm{d}x_n) \exp[-i\mathfrak{k}_2 x_2] ... \exp[-i\mathfrak{k}_n x_n] \cdot$$

$$\cdot \{\langle \Phi_0', \bar{\boldsymbol{O}}(\infty) T^*[\dot{\boldsymbol{O}}(x_2) ... \dot{\boldsymbol{O}}(x_n)]\Phi_0\rangle - \langle \Phi_0', T^*[\dot{\boldsymbol{O}}(x_2) ... \dot{\boldsymbol{O}}(x_n)]\bar{\boldsymbol{O}}(-\infty)\Phi_0\rangle\},$$

where

$$(2.27) \qquad \bar{\boldsymbol{O}}(\pm\infty) = \int_{-\infty}^{\infty} \boldsymbol{O}(\boldsymbol{x}, \pm\infty)\, \mathrm{d}\boldsymbol{x}\,.$$

The equation (2.26) shows the fact that *the soft bosons are always produced by the external lines of the total Feynman diagrams.* This has been shown by BROWN and FEYNMAN in a special case and by JOSEPH [7] in the case of the production of n photons.

Equation (2.26) further shows that the only soft boson whose mass is extremely small compared to the linear dimension of the source function can be produced as *a difference of the proper fields* before and after the collision. We will come back to this problem in Sect. **4**.

3. – Statistical Theory of the Boson Production (+).

If the number of the produced bosons is very large, we can naturally introduce the idea of statistics. This is the aim of the discussion in this section.

First, we will rewrite the general formula (2.24) in terms of the occupation

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(*) In a rigorous sense, the Compton wave length μ^{-1} of a meson must be regarded to be large compared to that of the source.

(7) L. M. BROWN and R. P. FEYNMAN: *Phys. Rev.*, **85**, 231 (1952); J. JOSEPH: *Ph. D. Thesis*, State University of Iowa (1955); J. M. JAUCH and F. ROHRLICH: *Theory of Photons and Electrons*, p. 390 (1956).

(+) See ITO's paper to be published in the *Progress of Theoretical Physics*. The author would like to thank Dr. ITO for his kind correspondence.

number of bosons and then determine the most probable number of bosons by taking variation with respect to the boson occupation number. It is worthwhile noticing that the discussion in this form will be done analogously to the determination of temperature in statistical mechanics. One of the advantages of the formulation along this line is that the relation between Fermi's and the LOW theories can be manifestly seen. This comparison will be made in the next section.

In order to apply the statistical idea, we shall introduce a large box V in which all reactions take place and the boson momentum $\mathfrak{k}$ is considered to be a discrete variable rather than a continuous one. Since the function $\varphi_p(\mathfrak{k}_1, \ldots, \mathfrak{k}_n)$ is symmetric with respect to momenta $\mathfrak{k}_1, \ldots, \mathfrak{k}_n$, we shall introduce new independent variables $n_1, n_2, \ldots$ instead of the original $\mathfrak{k}_1, \ldots, \mathfrak{k}_n$, where n_1 is the number of bosons with the momentum $\mathfrak{k}_{(1)}$, n_2 with $\mathfrak{k}_{(2)}$ and so on. In doing so, we can express φ_p by $(n_1, n_2, \ldots)$ as

$$(3.1) \qquad \varphi_p(\mathfrak{k}_1, \ldots, \mathfrak{k}_n) \equiv \xi_n(n_1, n_2 \ldots)$$

and replace the integration by summation as

$$(3.2) \qquad \int \mathrm{d}\boldsymbol{k} \rightarrow \frac{(2\pi)^3}{V} \sum .$$

By putting (3.1) and (3.2) into (2.24), we have

$$(3.3) \qquad w_n = (n!)^{-1} \prod_{i=1}^{n} \left\{ \sum_{k_i} \frac{1}{V} \frac{1}{2K_i^3} \right\} \cdot |\varphi_p|^2 = \sum_{n=n_1+n_2+\ldots} \prod_{s=1}^{\infty} \frac{1}{n_s!} \left\{ \frac{1}{V} \frac{1}{2K_{(s)}^3} \right\}^{n_s} \cdot |\xi_n|^2 ,$$

with

$$n = \sum^{\infty} n_s \qquad \text{(the total number of produced bosons).}$$

If we put

$$(3.5) \qquad \mathrm{d}w_n(n_1, n_2, \ldots) \equiv \prod_{s=1}^{\infty} \frac{1}{n_s!} \left\{ \frac{1}{V} \frac{1}{2K_{(s)}^3} \right\}^{n_s} \cdot |\xi_n(n_1, n_2, \ldots)|^2 ,$$

the total probability is

$$(3.6) \qquad w_n = \sum_{n=n_1+n_2+\ldots} \mathrm{d}w_n(n_1, n_2 \ldots) .$$

We shall now approximate w_n by the most probable $\mathrm{d}w_n$ i.e.,

$$(3.7) \qquad w_n \doteqdot \mathrm{d}w_n^{(\max)}(n_1, n_2, \ldots) .$$

The most probable probability $dw_n^{(\max)}$ will be easily obtained as follows. Introducing

$$I = \log dw_n(n_1, n_2, \dots,) + \beta(\Delta E - \sum K_{(s)} n_s) \, , \quad (*) \tag{3.8}$$

we will use the Lagrange method of maximizing I subjet to the energy conservation law, i.e.,

$$\frac{\delta I}{\delta n_s} = 0 \, . \tag{3.9}$$

From (3.5) we have

$$O = \frac{\delta I}{\delta n_s} = -\log n_s + \log \left\{ \frac{1}{V} \frac{1}{2K_{(s)}^3} \right\} - \frac{\delta}{\delta n_s} \log |\xi_n(n_1, n_2, \dots)|^2 - \beta K_{(s)} \, , \tag{3.10}$$

where we have made use of Stirling's formula

$$\frac{\partial}{\partial n_s} \log n_s! = \log n_s \, . \tag{3.11}$$

(3.10) is generally an infinite system of simultaneous equations. Since we cannot proceed our further general discussion without any assumption on ξ_n, we shall introduce the following simplification (+); $\varphi_p(\mathfrak{k}_1, \dots, \mathfrak{k}_n)$ *can be factorized*, i.e.,

$$|\varphi_p(\mathfrak{k}_1, \dots, \mathfrak{k}_n)|^2 = \varphi(\mathfrak{k}_1) \dots \varphi(\mathfrak{k}_n) |v(\Delta \boldsymbol{P}, \Delta E)|^2 \, , \tag{3.12}$$

where $|v(\Delta \boldsymbol{P}, \Delta E)|^2$ is the rest of the factorized part and, for instance, expresses a matrix element of elastic scattering. It does not generally depend on only the difference of initial and final energy-momenta but depends on them separately; $\Delta \boldsymbol{P}$, ΔE should be understood symbolically.

(*) As a matter of fact, we must also take into account the momentum conservation law. This will not be discussed here, since the generalization is quite straigthforward. The conservation of the total number n must not be considered, since we are interested in the most probable probability among all the possible n.

(+) The factorization means the mesons are produced in a statistically independent way. The statistical correlation will be discussed at some other occasion. In the case of low energy photons, the factorization of ξ_n has been proved. See reference (7).

From (3.12), we get

$$|\xi_n(n_1, n_2 \ldots)|^2 = \prod_s \{\varphi(\mathfrak{k}_{(s)})\}^{n_s} \cdot |v(\Delta \boldsymbol{P}, \Delta E)|^2 . \tag{3.13}$$

Under this assumption the equation (3.10) can be remarkably simplified and we have from (3.10)

$$n_s = \frac{1}{V}\,\Omega(\mathfrak{k}_{(s)}) \exp[-\beta K_{(s)}] , \tag{3.14}$$

where

$$\Omega(\mathfrak{k}_{(s)}) = \varphi(\mathfrak{k}_{(s)})/2K^3_{(s)} . \tag{3.15}$$

We will refer to (3.15) as « the volume of boson cloud » since it has the dimension of a volume.

The parameter β in (3.8) will be determined by energy-conservation law as follows,

$$\Delta E = \sum_s K_{(s)} n_s = \frac{1}{V} \sum_s K_{(s)} \Omega(\mathfrak{k}_{(s)}) \exp[-\beta K_{(s)}] = \tag{3.16}$$

$$= \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, K\Omega(\mathfrak{k}) \exp[-\beta K] .$$

The number of the produced mesons is

$$n = \sum_s n_s = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\, \Omega(\mathfrak{k}) \exp[-\beta K] . \tag{3.17}$$

If we eliminate β from the equations (3.16) and (3.17), we can express the number n in terms of ΔE, namely the multiplicity of the produced mesons.

It is immediately noticed that the discussion given above is developed quite analogously to the determination of the temperature of a many-particle-system in thermal equilibrium. This fact suggests that Fermi's thermodynamic approach would be verified by field theoretical considerations.

We will discuss the connection between Fermi's and field theoretical approaches in the following section.

4. – Discussion and Comparison of Theories.

4·1. *Fermi's Theory of Multiple Meson Production.* – This theory belongs to the group of the « excited field theories » according to Lewis' classification (1).

A volume Ω in which the energy transformations are supposed to take place in the course of the nucleon collision plays an essential role, and is the only adjustable parameter in the theory. Its choice is entirely arbitrary provided that it is physically reasonable.

Let us review briefly the essential structure of Fermi's theory.

The probability of n-meson production is given, generally speaking, by

$$w_n \sim \int \dots \int \mathrm{d}\boldsymbol{k}_1 \dots \mathrm{d}\boldsymbol{k}_n |H|^2 \varrho(k_1, \dots, k_n) , \tag{4.1}$$

where

$$\varrho(\boldsymbol{k}_1, \dots, \boldsymbol{k}_n)\, \mathrm{d}\boldsymbol{k}_1 \dots \mathrm{d}\boldsymbol{k}_n = \left\{ \frac{V}{(2\pi)^3} \right\}^n \mathrm{d}\boldsymbol{k}_1 \dots \mathrm{d}\boldsymbol{k}_n , \tag{4.2}$$

is the final density. FERMI assumes H ^{2} to be independent of the meson momenta and simply proportional to

$$(\Omega/V)^n . \tag{4.3}$$

Then we have

$$w_n \sim \int \dots \int \left\{ \frac{\Omega}{(2\pi)^3} \right\} \mathrm{d}\boldsymbol{k}_1 \dots \mathrm{d}\boldsymbol{k}_n , \tag{4.4}$$

which has a form as if the reactions take place in the box Ω. In order to treat the thermal equilibrium in the box Ω, the standard method of statistical mechanics will be used. FERMI does not explicitely write down equations of elementary statistical mechanics, but we shall reconstruct his theory to compare to the field theoretical discussion given in the previous section.

First, expressing (4.4) in terms of the occupation number of mesons and taking the most probable distribution by variation subject to the energy conservation law, we get the most probable number of mesons with momentum $\boldsymbol{k}$

$$n_s = \frac{\Omega}{V} \exp[-\beta K_s] \ (*). \tag{4.5}$$

The total number of the produced mesons is

$$n = \sum_s n_s = \frac{\Omega}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k} \exp[-\beta K] , \tag{4.6}$$

(*) Classical statistics has been taken for the sake of simplicity.

and the total energy is

$$(4.7) \qquad \Delta E = \sum_s K_{(s)} n_s = \frac{\Omega}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k} K \exp[-\beta K] \, .$$

Now, FERMI assumed

$$(4.8) \qquad \Delta E/\mu \gg K/\mu \gg 1 \, .$$

then (4.6) and (4.7) are, respectively (*),

$$(4.6') \qquad n = \frac{\Omega}{\pi^2} \beta^{-3} \, ,$$

$$(4.7') \qquad \Delta E = \frac{3\Omega}{\pi^2} \beta^{-4} \, .$$

From (4.6′) and (4.7′) we have

$$(4.9) \qquad n = (\tfrac{1}{3})^{\frac{3}{4}} (\Omega/\pi^2)^{\frac{1}{4}} (\Delta E)^{\frac{3}{4}} = 1.42 (\Delta E/M)^{\frac{3}{4}} / (E/M)^{\frac{1}{4}} \, ,$$

where we have used Fermi's choice that the volume Ω must be the Lorentz-contracted volume of a sphere with radius $R = \mu^{-1}$, i.e.,

$$(4.10) \qquad \Omega = \frac{4\pi}{3} \mu^{-3} \frac{2M}{E} \, .$$

It is now clear that Fermi's discussion is analogous to that of the previous section and Fermi's Ω is nothing but the « volume of meson cloud » introduced by field theoretical considerations. We see the following correspondence:

$$(4.11) \qquad \begin{cases} (3.14) - (4.5), \\ (3.17) - (4.6),\ (4.6') \, , \\ (3.16) - (4.7),\ (4.7') \, , \\ (3.15) - (4.10) \, . \end{cases}$$

A question arises as to what condition or what approximation gives the « constant volume of meson cloud (4.10) » in field theory. We cannot give any clear answer to this question at the present moment, but the following consideration would be useful to see the meaning of the « constant volume ».

(*) See Appendix II.

As we shall see in the next subsection, the field theoretical volume of cloud in the LOW theory shows

$$\Omega(\mathfrak{k}) = \frac{1}{2K^3}\varphi(\mathfrak{k}) = \Omega_0\left(\frac{\mu}{K}\right), \tag{4.12}$$

where Ω_0 is a constant. This is of course a Lorentz-contracted volume due to meson velocity in the cloud. If we assume that all the mesons collide with one another frequently in a course of scattering and the volume of the cloud is averaged over all possible energy, then

$$\overline{\Omega} = \frac{1}{\Delta E - \mu}\int_{\mu}^{\Delta E} \Omega(\mathfrak{k})\,\mathrm{d}K \simeq \Omega_0' \frac{\mu}{\Delta E}. \tag{4.13}$$

Here the fact is taken into account that the $\log \Delta E/\mu$ is a slowly varying function when $\Delta E \gg \mu$. (4.13) is approximately equal to (4.10). We must recall, however, that Fermi's choice (4.10) is not always satisfactory to explain experimental data, so it would not be right to try to derive (4.10) from field theory in an artificial manner.

4·2. *The LOW Theory* (1). – We shall discuss in this sub-section the relation between the LOW theory and our general theory.

As is well known, the LOW theory is a straightforward application of the Bloch-Nordsieck transformation in quantum electrodynamics, and is based on the idea that the mesons are produced as a difference of the meson proper field before and after the nucleon collision. This fact seems to be strange, because the meson proper field of a nucleon is attached close to the nucleon and the *difference* of attached fields is still attached field which would never be free mesons. As a matter of fact, if we calculate the cross section given by LOW, we will obtain zero cross section due to the energy conservation law. This point is clearly stated in our general formula (2.22) and (2.24).

In quantum electrodynamics, only the S-wave is produced as a difference of proper field due to the vanishing photon mass, as discussed by FIERZ and PAULI, but in quantum meson dynamics, the mesons are not produced as the difference of the fields as long as the meson mass is finite. In particular there exists no S-wave in PS(PV) theory so that the mesons are always produced not as a difference of the fields.

If we overlook the above mentioned point, the LOW theory can be derived by the general formula as follows.

Let us first assume the $\boldsymbol{O}$'s are commutable one another, as LOW do, then,

from (2.22), we have

$$\varphi(\mathfrak{k}_1, \ldots, \mathfrak{k}_n) = \left\langle \Phi_0', \prod_{i=1}^{n} \left\{ \int (\mathrm{d}x_i) \exp[-i\mathfrak{k}_i x_i] \dot{\boldsymbol{O}}(x_i) \right\} \Phi_0 \right\rangle \tag{4.14}$$

$$= \prod_{i=1}^{n} \left\{ \int \mathrm{d}\boldsymbol{x}_i \Delta \boldsymbol{O}(\boldsymbol{x}) \exp[-i\boldsymbol{k}_i \boldsymbol{x}_i \right\} v_{PQ} ,$$

here use has been made of the fact that all the mesons are of low energy and the one nucleon with momentum P is scattered by a potential v to the state of momentum Q. The meson mass is also neglected. $\Delta\boldsymbol{O}(\boldsymbol{x})$ is the difl ference between $\boldsymbol{O}(x_i)$ before and after the scattering. In the case of neutra- pseudo-scalar meson with pseudo-vector coupling,

$$\int \mathrm{d}\boldsymbol{x} \Delta \boldsymbol{O}(\boldsymbol{x}) \exp[-i\boldsymbol{k}\boldsymbol{x}] = f\Delta\boldsymbol{\sigma}\cdot\boldsymbol{k} , \tag{4.15}$$

where $\Delta\boldsymbol{\sigma}$ means the change of classical spin vector due to the scattering.

(4.15), (4.14) and (2.23) give

$$\mathrm{d}w_n = \prod_{i=1}^{n} \frac{\mathrm{d}\boldsymbol{k}_i}{(2\pi)^3} \frac{1}{2K_i^3} \{f\Delta\boldsymbol{\sigma}\cdot\boldsymbol{k}_i\}^2 \cdot |v_{PQ}|^2 . \tag{4.16}$$

In (4.16) the effect of the potential scattering is explicitly written as $|v_{PQ}|^2$. If we express (4.16) in terms of the occupation number of mesons

$$w_n = \sum \mathrm{d}w_n = \prod_s \frac{w_s^{n_s}}{n_s!} \cdot |v_{PQ}|^2 ,$$

with

$$w_s = \frac{1}{2K_{(s)}^3} \{f\Delta\boldsymbol{\sigma}\cdot\boldsymbol{k}_{(s)}\}^2 \frac{1}{V} .$$

Therefore the cloud volume is

$$\Omega(\mathfrak{k}_{(s)}) \equiv \frac{1}{2K_{(s)}^3} \{f\Delta\boldsymbol{\sigma}\cdot\boldsymbol{k}_{(s)}\}^2 . \tag{4.17}$$

We can now obtain the most probable distribution just like the discussion in Sect. **3**, and we obtain

$$\begin{cases} n = \dfrac{1}{(2\pi)^3} \displaystyle\int \mathrm{d}\boldsymbol{k} \Omega(\mathfrak{k}) \exp[-\beta K] = \dfrac{4\pi}{(2\pi)^3} \dfrac{f^2(\Delta\sigma)^2}{\beta^2} , \\ \Delta E = \dfrac{8\pi}{(2\pi)^3} \dfrac{f^2(\Delta\sigma)^2}{\beta^3} . \end{cases} \tag{4.18}$$

The multiplicity is

$$n = (f^2/8\pi^2)^{\frac{1}{3}} (\Delta\sigma)^{\frac{2}{3}} (\Delta E)^{\frac{2}{3}} , \tag{4.19}$$

which agrees to LOW's formula, as it should be. The above discussion shows that even the LOW theory could be written in a similar fashion to Fermi's thermodynamic approach, although the numerical results are inherent in each theory. The difference, of course, comes from the numerical difference of the volume of meson spectrum, namely

$$\Omega = \Omega_0 \frac{2M}{E} \qquad \text{in Fermi's theory}\,, \tag{4.20}$$

$$\Omega = \Omega_0' \frac{\mu}{K} \qquad \text{in the LOW theory}\,. \tag{4.20'}$$

It is of interest to notice that the factor μ/K in (4.20′) is the Lorentz-contraction factor due to mesons instead of the factor due to the incident nucleons in Fermi's intuitive choice.

Thus, we are able to interpret the field theory in terms of a language in statistical mechanics and vice versa, as long as the multiple boson processes are concerned.

4. – Conclusion and Discussion.

We have derived in the second section a general expression for the production of n mesons, which is useful to clarify the physical meaning of the process. Furthermore, if we introduce the occupation number representation and determine the most probable distribution, we will find the similarity between our discussion and the determination of the temperature in statistical mechanics. The equations determining the most probable state are generally infinite simultaneous equations (3.10), which is due to the mutual correlation of the produced mesons. All theories proposed so far are such that the meson correlation is entirely neglected, in which case the equation is remarkably simplified and reduced to (3.14). This shows a close relation between the « excited field theories » and the « pre-existing field theories », according to Lewis' classification (1), although it might not be useful for the actual computation of, say, the cross section of the processes.

* * *

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Appendix I

The General Formulas of χ_n for the Non-Linear Interaction.

When we derive the equation (2.12), the source $\boldsymbol{O}(x)$ is assumed to include no meson operator. We will discuss here the case

$$\boldsymbol{O}(x) \equiv \underbrace{\boldsymbol{U}(x) \ldots \boldsymbol{U}(x)}_{l}\boldsymbol{O}^{(0)}(x) \,, \tag{I.1}$$

where $\boldsymbol{O}^{(0)}$ no longer includes any meson operator. Let us denote

$$\underbrace{\boldsymbol{U}(x) \ldots \boldsymbol{U}(x)}_{i}\boldsymbol{O}^{(0)}(x) \,,$$

by $\boldsymbol{O}^{(i)}(x)$, namely

$$\boldsymbol{O}(x) = \underbrace{\boldsymbol{U}(x) \ldots \boldsymbol{U}(x)}_{l-i}\boldsymbol{O}^{(i)}(x) \equiv \boldsymbol{O}^{(l)}(x) \,, \qquad (i = 1, 2, \ldots, l). \tag{I.2}$$

In this case the operator (2.11) satisfies the following equation,

$$\begin{aligned} i\frac{\delta\chi_n(x;\tau)}{\delta\tau(x')} &= \frac{1}{n!}\sum_{k=0}^{n-1}\{U^{(+)}(x,\tau)\}^k\, i\,\frac{\delta U^{(+)}(x,\tau)}{\delta\tau(x')}\{U^{(+)}(x,\tau)\}^{n-k-1} \\ &= \frac{1}{n!}\sum_{k=0}^{n-1}\{U^{(+)}(x,\tau)\}^k i\Delta^{(+)}(x-x')\boldsymbol{O}(x')\{U^{(+)}(x,\tau)\}^{n-k-1} \\ &= \frac{1}{n!}\, i\Delta^{(+)}(x-x')\boldsymbol{O}^{(0)}(x')\sum_{k=0}^{n-1}\{U^{(+)}(x,\tau)\}^k\{U(x'/\tau)\}^l\{U^{(+)}(x,\tau)\}^{n-k-1} \\ &= \frac{1}{n!}\, i\Delta^{(+)}(x-x')\boldsymbol{O}^{(0)}(x')\sum_{k=0}^{n-1}\sum_{j=1}^{(k,l)}(j!)\binom{k}{j}\binom{l}{j}\{i\Delta^{(+)}(x-x')\}^j\cdot \\ &\qquad\cdot\{\boldsymbol{U}(x)\}^{l-j}\{U^{(+)}(x,\tau)\}^{n-j-1} = \\ &= \frac{1}{n!}\sum_{k=0}^{n-1}\sum_{j=1}^{(k,l)}(j!)\binom{k}{j}\binom{l}{j}\{i\Delta^{(+)}(x-x')\}^{(j+1)}\boldsymbol{O}^{(l-j)}(x')\{U^{(+)}(x,\tau)\}^{n-j-1} \\ &= \frac{1}{n!}\sum_{k=0}^{n-1}\sum_{j=1}^{(k,l)}(n-j-1)!(j!)\binom{k}{j}\binom{l}{j}\{i\Delta^{(+)}(x-x')\}^{(j+1)}\cdot \\ &\qquad\cdot\boldsymbol{O}^{(l-j)}(x')\chi_{(n-j-1)}(x;\tau) \,. \end{aligned} \tag{I.3}$$

$$\begin{aligned} \chi_n(x;\tau) = \frac{-i}{n!}\sum_{k=0}^{n-1}\sum_{j=0}^{(k,l)}(n-j-1)!(j!)\binom{k}{j}\binom{l}{j}\cdot \\ \cdot\int_{-\infty}^{\tau}(\mathrm{d}x')\{i\Delta^{(+)}(x-x')\}^{j+1}\boldsymbol{O}^{(l-j)}(x')\chi_{n-j-1}(x;\tau') \,, \end{aligned} \tag{I.4}$$

where (k, l) stands for the minimum of k and l, and the following lemma has been used: if $[A, B] = \lambda$ (c-number), then

$$A^n B^m = \sum_{j=0}^{(n,m)} (j!) \binom{n}{j} \binom{m}{j} (\lambda)^j B^{m-j} A^{n-j} , \tag{I.5}$$

here (n, m) again means the minimum of n and m. (I.4) can be solved by iteration, but the solution is generally quite complicated.

Appendix II

The Integrations (4.6) and (4.7) when μ/K is not Very Small.

$$\begin{aligned} I(\beta) &\equiv \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k} \exp[-\beta K] \\ &= 4\pi \int_0^{\infty} \mathrm{d}k \cdot k^2 \exp[-\beta\sqrt{k^2+\mu^2}] \\ &= 4\pi\mu^3 \int_0^{\infty} x^2 \,\mathrm{d}x \exp[-\beta\mu\sqrt{x^2+1}] , \\ &= 4\pi\mu^3 \int_0^{\infty} \mathrm{d}\theta \sinh^2\theta \cdot \cosh\theta \cdot \exp[-\beta\mu\cosh\theta] \\ &= \pi\mu^3 \int_0^{\infty} \mathrm{d}\theta \{\cosh 3\theta - \cosh\theta\} \exp[-\beta\mu\cosh\theta] \\ &= \pi\mu^3 \{K_3(\beta\mu) - K_1(\beta\mu)\} \\ &= 4\pi\mu^3 K_2(\beta\mu)/\beta\mu , \end{aligned} \tag{II.1}$$

where use has been made of

$$K_n(Z) = \int_0^{\infty} \mathrm{d}\theta \cosh n\theta \exp[-Z\cosh\theta] , \qquad (|\arg Z| < \tfrac{1}{2}\pi), \tag{II.3}$$

$$K_{n+1}(Z) - K_{n-1}(Z) = \frac{2n}{Z} K_n(Z) . \tag{II.4}$$

$K_n(Z)$ is Schläfli's Bessel function ([8]).

The number of the mesons is

$$n = \frac{\Omega}{(2\pi)^3} I(\beta) = \frac{\Omega}{2\pi^2} \mu^3 K_2(\beta\mu)/\beta\mu \,. \tag{II.5}$$

In a similar manner, the energy is

$$\Delta E = \frac{\Omega}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k} K \exp[-\beta K] = \frac{\Omega}{(2\pi)^3} \pi\mu^4 \{3K_3(\beta\mu) + K_1(\beta\mu)\}/\beta\mu \,. \tag{II.6}$$

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(8) G. N. WATSON: *Theory of Bessel Functions* (Cambridge, 1922), p. 181.

RIASSUNTO (*)

Si deriva una formula generale per la produzione multipla dei bosoni. Si introduce un concetto statistico nella formula generale. Non si eseguono confronti con dati sperimentali nè si interpretano questi ultimi; si restringe bensì la discussione alla trattazione delle diverse teorie finora proposte dal punto di vista dell'unificazione. Si discute dettagliatamente in modo particolare il fondamento che la teoria statistica di Fermi trova nella teoria dei campi.

(*) *Traduzione a cura della Redazione.*

Holes and Particles in Shell Models.

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Summary. - The study of nuclear and atomic shell models involves the calculation of matrix elements of operators between many-body configurations. This paper shows how the occupation number representation of Dirac can be used in such calculations, especially for configurations near closed shells. Methods for the construction of many-body wave functions are given, and the equivalence of holes and particles discussed. Application is made to the calculation of transition matrix elements for holes and particles and to the investigation of two body interactions. In particular the interaction between holes and particles is considered.

1. – Introduction.

When studying systems of fermions in problems of atomic or nuclear structure often it is necessary to evaluate matrix elements of sum operators between states differing from closed-shell configurations by a few particles. A consistent use of an occupation number representation (1) simplifies these calculations considerably. The evaluation of matrix elements reduces to the manipulation of creation and annihilation operators, thus providing a simple procedure for counting terms in which exchange contributions appear automatically. State vectors in the occupation number representation have simple normalisation properties and are already antisymmetrized.

2. – Occupation Number Representation.

We can take as a basis for a system of fermions the set of all antisymmetrized products of single particle wave functions. Alternatively we can introduce the numbers n_1, n_2, ... of fermions occupying the single particle

(1) P. A. M. Dirac: *Principles of Quantum Mechanics*, Chap. X (Oxford, 1947).

states $|1\rangle$, $|2\rangle$, and treat them as dynamical variables. They each have as eigenvalues 0 or 1, and together form a complete set of commuting variables for the assembly. The set of states diagonal in the n_a form a basis for the occupation number representation. A state in this set is determined by giving the values of the occupation numbers of the single particle states.

2·1. *Creation and annihilation operators.* – These basic states can be built up from the vacuum state by introducing the creation operator η_a and its adjoint, the annihilation operator, η_a^+ (²) for each single particle state $|a\rangle$. They have the simple commutation properties (¹)

$$(1)\qquad \begin{cases} \eta_a\eta_b + \eta_b\eta_a = 0 \\ \eta_a^+\eta_b^+ + \eta_b^+\eta_a^+ = 0 \\ \eta_a\eta_b^+ + \eta_b^+\eta_a = \delta_{ab}\,. \end{cases}$$

The occupation number of the state a is the eigenvalue of

$$(2)\qquad \eta_a\eta_a^+\,|a\rangle = n_a|a\rangle$$

and the matrix elements of η_a and η_a^+ are

$$(3)\qquad \langle n_a|\eta_a|n_a'\rangle = \langle n_a'|\eta_a^+|n_a\rangle = n_a(1-n_a')\,,$$

where, of course, the n_a can only take the values 0 or 1.

Further, let T be an operator which is a symmetric sum of single particle operators, $T=\sum_i t_i$, where t_i acts only on the i-th particle. In the occupation number representation this has the form (¹)

$$(4)\qquad T=\sum_{ab}\eta_a\eta_b^+\langle a|t|b\rangle\,,$$

where the number $\langle a|t|b\rangle$ is the matrix element of t taken between single particle states $|a\rangle$ and $|b\rangle$. Similarly, the two-body operator $V=\sum_{ij}v_{ij}$ becomes

$$(5)\qquad V=\sum_{abcd}\eta_a\eta_b\eta_d^+\eta_c^+\langle ab|v|cd\rangle\,.$$

When we have a mixed system, as in nuclei, of two different Fermi particles,

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(²) We define the adjoint A^+ of an operator A by

$$\langle a|A^+|b\rangle = \langle b|A|a\rangle^*$$

where * means complex conjugate.

there are two ways of proceeding. If we regard them as different states of the same fundamental particle, we adopt the isobaric spin formalism, and demand complete antisymmetry of the wave function with respect to exchange of i-spin, as well as space and ordinary spin, variable. Then the label on η includes the 3-component of i-spin for the state, the antisymmetry is ensured, and we proceed as before. However, if we regard them as distinct particles, we impose no symmetry requirement on the interchange of one type with the other. Then we must define two sets of creation operators, η and ξ, one for each type of particle, which commute with each other, $\eta\xi = \xi\eta$, while still obeying the commutation relations (1) among themselves. Other properties, such as rotational and vector coupling discussed below, remain the same.

2·2. *Coupled states and rotational properties.* – We are generally interested in states which are eigenfunctions of the angular momentum. We write the creation operator for a particle in a state with angular momentum quantum numbers (j, m) as η_{jm}; other quantum numbers which may be required to specify the state completely we shall not write explicitly. Then the one-particle state vectors are

$$|jm\rangle = \eta_{jm}|0\rangle\,, \tag{6}$$

where $|0\rangle$ is the vacuum state. Since the η_α have the same transformation properties as the corresponding states $|\alpha\rangle$, ([1]) η_{jm} behaves under rotations of the co-ordinate system as an irreducible tensor of rank j ([3]). We may then construct two-particle vectors in the usual way

$$|jj'JM\rangle = N \sum_m \eta_{jm}\eta_{j'M-m}|0\rangle\langle jmj'M-m|JM\rangle \tag{7}$$

where $\langle jmj'm'|JM\rangle$ is the usual vector addition coefficient ([4]). The normalization factor N is one unless the particles are in the same orbit $(j=j')$, when $N = (2)^{-\frac{1}{2}}$.

2·3. *Fractional parentage.* – We may go further and construct states for n-particle systems by operating with a creation operator on the vectors for $n-1$ particles, and taking suitable linear combinations. (7) is a trivial example of this. Consider now n particles in the same orbit j in some symmetry arran-

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([3]) M. E. Rose: *Multipole Fields* (New York, 1955). An irreducible tensor $T(k)$ of rank k is a set of $(2k+1)$ components T_{kq} which transform under co-ordinate rotations as

$$T'_{kq} = RT_{kq}R^{-1} = \sum_{q'} T_{kq'}D^k_{q'q}(R)\,.$$

Then $U_{kq} = (-)^{k+q}T^+_{k-q}$ is also a tensor component in this sense.

([4]) E. U. Condon and G. H. Shortley: *Theory of Atomic Spectra* (Cambridge, 1935).

gement $\alpha(n)$ with total angular momentum (J, M). Using (2), the total number in the orbit j is given by

$$n = \sum_m n_m = \sum_m \eta_{jm}\eta^+_{jm} ,$$

so that

$$(8) \qquad n\,|\alpha(n)JM\rangle = \sum_m \eta_{jm}\eta^+_{jm}\,|\alpha(n)JM\rangle$$
$$= \sum_{m\beta p} \eta_{jm}\,|\beta(n-1)_p M_p\rangle\langle\beta(n-1)J_p M_p\,|\eta^+_{jm}\,|\alpha(n)JM\rangle$$
$$= \sum_{m\beta p} \eta_{jm}\,|\beta J_p M_p\rangle\langle\beta J_p\|\eta^+_j\,\|\alpha J\rangle\langle J_p M_p jm\,|JM\rangle\,.$$

In the second step we have expanded into intermediate states which are just the parent states of $n-1$ particles in arrangements $\beta(n-1)$. The last step uses the Wigner-Eckart theorem (5).

Apart from normalization, the reduced (or double-barred) matrix elements are the coefficients of fractional parentage (6)

$$(9) \qquad \langle\beta(n-1)J_p, j\}\alpha(n)J\rangle = n^{-\frac{1}{2}}\langle\beta(n-1)J_p\|\eta^+_j\,\|\alpha(n)J\rangle\,.$$

2·4. *Closed shells and the hole concept.* – A shell or orbit is said to be closed when all $2j+1$ single particle magnetic substates, $j \geqslant m \geqslant -j$, are occupied. It is then a spherically symmetric system with zero angular momentum, represented by a single vector $|n_{jm}=1$, all $m\rangle$, which we may denote $|C\rangle$. The addition of one extra particle, outside the closed shell, in the single particle orbit $|j'm'\rangle$ then gives the resultant state $|(C+1)j'm'\rangle = \eta_{j'm'}|C\rangle$. On the other hand, a shell j filled but for one particle necessarily has a resultant angular momentum $(j\,m)$, and is equivalent to the absence of a particle from the state $|j, -m\rangle$. This we can represent by the vector $|(C-1)j\,m\rangle = -(-)^{j+m}\eta^+_{j-m}|C\rangle$, and refer to as a hole with angular momentum (jm). The phase $(-)^{j+m}$ is chosen to give the correct rotational properties (3).

Clearly, then, the state of a system with some closed shells (denoted by C) plus n particles in some arrangement $w(n)$ may be written in the occupational number representation as the product vector

$$(10) \qquad |C+w(n)\rangle = |C\rangle\,|w(n)\rangle\,,$$

where $|w(n)\rangle$ is built up from single particle vectors as described in 2.1 and 2.2. We wish to find a similar way of describing a system with the shells C closed

(5) G. Racah: *Lectures on Group Theory* (Princeton, 1951). We use the normalization $\langle jm\,|T_{kq}\,|j'm'\rangle = \langle jm\,|j'm'kq\rangle\langle j\|T_k\|j'\rangle$.

(6) G. Racah: *Phys. Rev.*, **63**, 367 (1943).

but for n vacancies or holes. Now any occupation number vector can be equally well regarded as describing the number of holes in each state; one particle = zero holes, zero particles = one hole. So that the superposition of such vectors describing the arrangement $|w\rangle$ of $C-n$ particles can also be understood as describing the conjugate state $|\bar{w}\rangle$ (in the sense of RACAH [7] and reference [4]) of n holes. The exact relation between such conjugate states does not concern us, as the hole states form a complete set and may be used throughout. Then we have to regard $(-)^{j+m}\eta^{+}_{j-m}$ as *creating* a hole with angular momentum (j, m), and η_{j-m} as *annihilating* it, so that the basic single-hole states are

$$|\bar{jm}\rangle = (-)^{j+m}\eta^{+}_{j-m}|\bar{C}\rangle . \tag{11}$$

With these conventions we can immediately write for $C-n$ particles the analog of (10)

$$|\bar{C}+\bar{w}(n)\rangle = |\bar{C}\rangle|\bar{w}(n)\rangle , \tag{12}$$

where $\bar{C}$ refers to all completely *unoccupied* shells and $\bar{w}(n)$ is built up from states (11).

3. – Applications.

3·1. *One-body operators.* – As a first simple application, we consider the one-body operator T of (4). For $C+n$ particles, the matrix element of T between two states (10) specified by $w_1(n)$ and $w_2(n)$ can be split into two parts

$$\langle 1|T|2\rangle = \sum_\gamma \langle\gamma|t|\gamma\rangle\,\delta(w_1, w_2) + \sum'_{\alpha\beta}\langle w_1|\eta_\alpha\eta^{+}_\beta|w_2\rangle\langle\alpha|t|\beta\rangle . \tag{13}$$

The first sum is over all single particle states $|\gamma\rangle$ in the closed shells, and vanishes unless (i) the outer n-particle states are the same, and (ii) T is a scalar. The latter follows from the spherical symmetry of closed shells. The second sum is over the states outside C only. Expanding this

$$\langle w_1(n)|\eta_\alpha\eta^{+}_\beta|w_2(n)\rangle = \sum_p \langle w_1(n)|\eta_\alpha|w_p(n-1)\rangle\langle w_p(n-1)|\eta^{+}_\beta|w_2(n)\rangle$$

and using (9) immediately gives the familiar fractional parentage expression [6].

For $C-n$ particles, thought of as n holes, η and η^{+} interchange roles. Using (12) and the commutation relation (1) we get the analog of (13)

$$\langle 1|T|2\rangle = \sum_\gamma \langle\gamma|t|\gamma\rangle\,\delta(\bar{w}_1, \bar{w}_2) - \sum'_{\alpha\beta}\langle\bar{w}_1|\eta^{+}_\beta\eta_\alpha|\bar{w}_2\rangle\langle\alpha|t|\beta\rangle \tag{14}$$

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(7) G. RACAH: *Phys. Rev.*, **62**, 438 (1942).

where, however, the second sum has changed sign and α, β have reversed order. $\langle\alpha|t|\beta\rangle$, of course, is still the single *particle* matrix element.

To illustrate further let us consider the case $n = 1$, so we may compare matrix elements for a single hole and a single particle. The reduced matrix elements (5), if T is a tensor of rank k, become, if $j_1 \neq j_2$,

$$\langle C+j_1\|T_k\|C+j_2\rangle = \langle j_1\|t_k\|j_2\rangle \tag{15}$$

and

$$\langle C-j_1\|T_k\|C-j_2\rangle = (-)^{k+j_1-j_2+1}\left[\frac{2j_2+1}{2j_1+1}\right]^{\frac{1}{2}}\langle j_2\|t_k\|j_1\rangle\,. \tag{16}$$

It is of interest to note that if $T^{+}(k)$ is the emission operator for 2^k-pole electromagnetic radiation, the hole element (16) reduces to that for the particle, (15), so the transitions have identical probabilities [8]. In general, if T is such that $T_{kq} = (-)^{k-q-p}T_{k-q}$, (15) and (16) differ only by a phase $(-)^{p-1}$. In particular, if t is the scalar $l\cdot s$, so $k = 0$, $j_1 = j_2$, we get the well-known result that a hole has spin-orbit coupling of opposite sign to a particle.

Another example of interest is the matrix element of T between a closed shell state, and the state in which one of the particles has been excited to another shell. It is simplest here to start from first principles. The particle + hole state is given by operating on the closed shell vector $|C\rangle$ according to the rules of Part 2. We use a bar, $\bar{\ }$, to denote quantum numbers of holes.

$$|\bar{j}j_1JM\rangle = \sum_m (-)^{j-m}\eta^+_{jm}\eta_{j_1M+m}|C\rangle\langle j-mj_1M+m|JM\rangle\,. \tag{17}$$

The matrix element is then

$$\langle(C)\,00|T_{kq}|\bar{j}j_1JM\rangle = \sum_{mab}(-)^{j-m}\langle a|t_{kq}|b\rangle\langle j-mj_1M+m|JM\rangle\cdot$$
$$\cdot\langle C|\eta_a\eta_b^+\eta_{jm}^+\eta_{j_1M+m}|C\rangle\,.$$

Using (1) and (2), the last factor is just $-\delta(j, j_a)\,\delta(m, m_a)\,\delta(j_1, j_b)\,\delta(M+m, m_b)$ and the reduced matrix element becomes

$$\langle(C)0\|T_{kq}\|\bar{j}j_1J\rangle = \sqrt{2j+1}\,\langle j\|t_k\|j_1\rangle\,\delta(J, k)\,. \tag{18}$$

Comparing this with the matrix element between the *two-particle* states $|jj_1J\rangle$

(8) G. R. SATCHLER: *Proc. Phys. Soc.* (*London*), A **67**, 1024 (1954), where $T^{+}(L)$ is denoted simply as L. The sign of the matrix elements for electric radiation depends upon the sign of $E_1 - E_2$, so it must be remembered that the energies of the *particle* states on the right of (16) are in an order inverse to the corresponding *hole* states.

and $|j^2 0\rangle$ [8]

(19) $$\langle j^2 0 \| T_k \| j j_1 J\rangle = \sqrt{2}\,\langle j \| t_k \| j_1\rangle\, \delta(J, k)\,,$$

we see the particle-hole transition is enhanced in amplitude by $[\frac{1}{2}(2j+1)]^{\frac{1}{2}}$.

3·2. *Two-body operators.* – We now consider operators of the type (5), the most important, of course, being the two-body interaction potential, $\frac{1}{2}\sum_{i\neq j} v_{ji}$: States like (10) of n particles in addition to closed shells give matrix elements

$$\langle C+w_1(n)|V|C+w_2(n)\rangle = \sum_{abcd}\langle C+w_1|\eta_a\eta_b\eta_d^+\eta_c^+|C+w_2\rangle\langle ab|v|cd\rangle\,.$$

The sum over a, b, c, and d divides into three sets of terms, (i) all indices refer to states in C, so either $a=c$, $b=d$, or $a=d$, $b=c$, and $w_1=w_2$ for non-zero contributions, (ii) only one pair refer to states in C, so w_1 and w_2 may differ at most in one single particle state, and (iii) none refer to C, hence w_1 w_2 may differ in two single-particle states. Then we can write

(20) $$\tfrac{1}{2}V = V(C) + \sum_i' u_i + \tfrac{1}{2}\sum_{ij}' v_{ij}\,,$$

where the prime on $\sum$ means sum over particles outside closed shells only. The number $V(C)$ is the total interaction within the closed shells,

$$V(C) = \tfrac{1}{2}\sum_{ab}[\langle ab|v|ab\rangle - \langle ab|v|ba\rangle]\,,$$

summed over all states a, b in the closed shells, and the last term in (20) is just the interaction between particles outside closed shells. The one-body operator u_i is the interaction of the i-th outer particle with the closed shells

$$\sum_i' u_i = \sum_{ab}' \eta_{ia}\eta_b^+\langle a|u|b\rangle$$

$$\langle a|u|b\rangle = \sum_c[\langle ac|\bar{v}|bc\rangle - \langle ac|\bar{v}|cb\rangle]\,,$$

summed over all states c in the closed shells. $\bar{v}$ denotes the symmetrized operator $\frac{1}{2}(v_{12}-v_{21})$; if v_{ij} is already symmetric, this is unnecessary. If v is scalar, so is u, and $|a\rangle$, b can differ at most in their radial dependence; when v_{ij} is the two-body interaction and the single particle states are chosen self-consistently, the $a\neq b$ terms will compensate corresponding off-diagonal terms of the kinetic energy operator.

We obtain similar results with almost closed shells, ($C-n$ particles), when regarded as states of n holes like (12). Again, η, η^+ interchange roles when applied to hole states, with the phase convention (11). Proceeding as before

and collecting terms we find

$$\tfrac{1}{2}V = V(C) - \textstyle\sum_i' u_i + \tfrac{1}{2}\sum_{ij}' v_{ij}\,, \tag{21}$$

where now $\sum'$ means sum over *holes* only. $V(C)$ and u_i are defined as before. As expected [4], u_i now occurs with opposite sign, since $V(C)$ includes contributions from states which are not occupied now. However, for more than one hole, $\sum u_i$ overcompensates for this, since it is the interaction of one particle with the *complete* shell. Then it subtracts *twice* the interactions between vacant states, so the last term, in v_{ij}, retains its positive sign.

The simplicity of the present procedure is to be contrasted with the more conventional one [4-6].

3·3. *Interaction between holes and particles.* – As a final application we shall consider the matrix elements of the two-body operator (5) within a system of C particles in which there is one vacancy or hole in the closed shells, and one particle in an orbit outside. We call this the interaction between a hole and a particle.

The state vectors have already been defined in (17). Proceeding as before with the matrix element of V, we separate the sum over states a, b, c and d, and find we can write

$$\tfrac{1}{2}V = V(C) - u_1 + u_2 - i_{12}\,, \tag{22}$$

$V(C)$, u_1, and u_2 have the same significance as in (20) and (21), namely the closed shells interaction energy, and binding energies of the missing particle and extra particle, respectively, and are diagonal in the angular momentum quantum numbers. i_{12} is the interaction between the odd particle and the hole (whose quantum numbers are distinguished by a bar), with matrix elements

$$\begin{aligned} &\langle \bar{j}_1 j_2 J \,|\, i_{12} \,|\, \bar{j}'_1 j'_2 J\rangle = \\ &= \textstyle\sum_{mm'} (-)^{j_1 - m + j'_1 - m'} \langle JM \,|\, j_1 - m\; j_2 M+m\rangle \langle JM \,|\, j'_1 - m'\; j'_2 M+m' \;\cdot \\ &\cdot [\langle j'_1 m'\; j_2 M+m \,|\, v \,|\, j_1 m j'_2 M+m'\rangle - \langle j' m'\; j_2 M+m \,|\, v \,|\, j'_2 M+m' j_1 m\rangle] = \\ &= (-)^{j_1 + j'_1 + j_2 + j'_2} \textstyle\sum_k (2k+1) W(j'_2 j_1 j'_1 j_2;\, kJ) \langle j'_1 j_2 k \,|\, v_{12} \,|\, j_1 j'_2 k\rangle\,, \end{aligned} \tag{23}$$

where we must use antisymmetrized two-particle states in the last matrix element. In a mixed system (as in nuclei), when we consider the two types of particles as distinct, and the vacancy is of one type, the extra particle of the other, the exchange terms do not appear in (23). We may treat both cases together by taking only the direct terms, and then replacing v by $v(1 - P^\sigma P^x)$

when particle and hole are of the same type. This automatically introduces the exchange term.

Changing to L–S coupling and using symmetry properties of the vector addition coefficients, we can write

$$(24)\qquad \langle \bar{j}_1 j_2 J \,|\, i_{12} \,|\, \bar{j}'_1 j'_2 J\rangle = \sum_{LS} \langle J j_1 j_2 \,|\, L l_1 l_2 S\rangle \langle J j'_1 j'_2 \,|\, L l'_1 l'_2 S\rangle \langle L \bar{l}_1 l_2 S \,|\, i_{12} \,|\, L \bar{l}'_1 l'_2 S\rangle\ .$$

Now the general central interaction is

$$v_{12} = J(r_{12})[w + mP^x + bP^\sigma - hP^\sigma P^x]\,,$$

in each term of which the spin and spatial operators appear as simple products. We find the spin contributions to the L–S matrix elements in (24) to be simple: for spin independent operators, they are unity, independent of S, and for spin exchange they have the value 2 in singlet states ($S = 0$) and vanish in triplet ($S = 1$). Also, without space-exchange P^x, the orbital contributions are identical to the corresponding two *particle* interaction. Thus for Wigner forces, particle hole interactions are the same as two-particle interactions, and for Bartlett forces, they become the same if we replace P^σ by $1 - P^\sigma$ in v.

There is no such correspondence for the space exchange operator P^x, but the orbital matrix elements have the simple form

$$\langle L\bar{l}_1 l_2 \,|\, J(r)P^x \,|\, L\bar{l}'_1 l'_2\rangle = [(2l_1+1)(2l_2+1)(2l'_1+1)(2l'_2+1)]^{\frac{1}{2}}(2L+1)^{-2}\cdot$$
$$\cdot\langle L0 \,|\, l_1 0 l_2 0\rangle\langle L0 \,|\, l'_1 0 l'_2 0\rangle R^L(l'_1 l_2 l'_2 l_1)$$

where R^L is the Slater radial integral [9]. This reduces to the result of SHORTLEY and FREED [10] when $l_1 = l'_1$ and $l_2 = l'_2$.

(9) I. TALMI: *Helv. Phys. Acta*, **25**, 185 (1952).
(10) G. H. SHORTLEY and B. FREED: *Phys. Rev.*, **54**, 739 (1938).

RIASSUNTO (*)

Lo studio di modelli nucleari e atomici a gusci richiede il calcolo di elementi di matrice di operatori tra configurazioni di più corpi. Il presente lavoro mostra come in tali calcoli possa usarsi la rappresentazione per mezzo dei numeri d'occupazione di Dirac, specialmente per configurazioni prossime agli strati chiusi. Si danno metodi per la costruzione di funzioni d'onda di più corpi e si discute in modo particolare l'equivalenza tra buche e particelle. Si fa un'applicazione al calcolo degli elementi di matrici di transizione fra buche e particelle e all'esame dell'interazione fra due corpi. Si considera in particolare l'interazione fra buche e particelle.

(*) *Traduzione a cura della Redazione.*

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Matter and Anti-Matter.

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Summary. — The implications of anti-matter (consisting of positrons and antiprotons) existing on a cosmic scale are discussed. It is shown from a consideration of the kinetic and magnetic energy of the interstellar gas clouds and from the energy of cosmic radiation that the ratio of anti-matter to ordinary matter in our Galaxy cannot exceed $\sim 10^{-7}$. The importance to radio astronomy of such a limit being attained is investigated. It appears that the intensities of radio emission from galactic and extra-galactic sources may be quantitatively explained on this hypothesis. The relation of these ideas to cosmology is also briefly discussed.

1. – Anti-Matter in our Own Galaxy: An Upper Limit.

The discovery of the anti-proton (1) raises a number of astrophysical and cosmological questions that will eventually have to be taken seriously if a complete symmetry of matter and anti-matter is established beyond all doubt. At the present time, from the point of view of elementary particle theory there is no reason to question the symmetry postulate.

It is therefore of some interest to ask whether anti-stars and anti-galaxies may exist. A consideration of these questions reveals the disturbing situation that even if the material of the universe were in actuality distributed symmetrically between what we might describe as the + and — forms, such a state of affairs would not be revealed by the methods so far used in optical astronomy. It is the purpose of this article to point out that some information

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(1) O. Chamberlain, E. Segrè, C. Wiegand and T. Ypsilantis: *Phys. Rev.*, **100**, 947 (1955).

can be deduced, however, from the known gas motions within our Galaxy, and also from considerations of radio astronomy. In particular, it seems possible to set a limit to the ratio of + and — in the interstellar gas of our own Galaxy.

Let n_+, n_- be the average number densities per cm³ of + and — in the interstellar gas of the Galaxy. The annihilation cross-section between the two forms may be taken (at the low temperature of the interstellar gas) as about $10^{-25}c/v$ cm², where v is the relative velocity of collision. Hence the annihilation rate per cm³ is of the order of $10^{-25}\,cn_+n_-$, and the energy production per cm³ per s is of the order of $10^{-17}n_+n_-$ erg.

The annihilation products would ultimately be γ-rays, neutrinos, and electrons and positrons. The γ-rays travel so far ($> 10^{24}$ cm) before giving rise to an electron-positron pair, that their energy, together with that carried by the neutrinos, must be lost from the Galaxy. Probably about 10 percent of the annihilation energy will be carried by the electrons (*), which individually receive energies of the order of 100 MeV (+). At these energies the electrons lose their energy mainly through collisional ionization and excitation, and through bremsstrahlung. The contribution of synchrotron emission in the galactic magnetic field is comparatively small if the magnetic intensity is taken to have the currently accepted value of about 10^{-5} gauss. Also for the velocities usually taken as applicable to the interstellar clouds (~ 10 km/s), the energy gained by the electrons in collisions with the clouds, by the Fermi mechanism (2), can be neglected. On this basis, and with a number density of hydrogen atoms of 1 per cm³, the lifetime of the electrons is found to be of the order of 10^{14} s.

The energy production per cm³ per second in the form of high speed electrons is about $10^{-18}n_+n_-$ erg. If this energy production were maintained over the lifetime of the electrons, the energy density of electrons would be built up to about $10^{-4}n_+n_-$ erg per cm³. Now this value cannot exceed $\sim 10^{-11}$ erg per cm³, which is approximately the total energy density in the interstellar medium of magnetic flux, turbulent kinetic energy, and cosmic radiation, otherwise the pressure exerted by the electrons would induce violent motions and heating in the interstellar gas, motions and heating that would be appreciably greater then is actually observed. Such a reaction on the interstellar gas would take place mainly via the magnetic field which would be

(*) In the remainder of this paper both electrons and positrons will be designated electrons.

(+) The relative probabilities of the production of 2 to 5 mesons in an annihilation process have been calculated by Belenky, Nikitov and Rosen, and by Sundarshan, and some results were reported at the Rochester Conference on High-Energy Physics, April 1956.

(2) E. Fermi: *Phys. Rev.*, **75**, 1169 (1949).

disturbed by the electrons and which in turn would disturb the interstellar gas. These arguments yield the condition that $n_- \leqslant \sim 10^{-7}(n_+)^{-1}$, or if $n_+ = 1$, $n_- \leqslant \sim 10^{-7}$. Accordingly it seems unlikely that the density of — in the interstellar gas can be much more than about one part in 10^7.

At first sight it might be thought that a more stringent upper limit could be obtained from the known observed properties of cosmic rays incident on the Earth, but this does not seem to be so. It is emphasized that the — protons move only at thermal speeds and hence possess very small magnetic rigidity. MEREDITH, VAN ALLEN, and GOTTLIEB (3) (c.f. also the work of NEHER, PETERSON, and STERN (4)) have reported the absence of primary cosmic rays with magnetic rigidity less than $1.7\cdot10^9$ eV. Moreover if, as seems likely, this absence of low energy primaries is due, not to a natural cut-off in the energy spectrum, but to shielding by local magnetic fields within the solar system, this same shielding must prevent electrons of 100 MeV from reaching the Earth.

The question now arises as to how a small density of — might conceivably have found its way into the Galaxy. This question demands an answer since the lifetime against annihilation for an anti-atom is only about $3\cdot10^{14}$ s if $n_+ = 1$, so that manifestly any — initially present some 10^{17} s ago at the time of formation of the Galaxy would have long ago disappeared. Possible answers to this question will be discussed in our concluding paragraph. We shall suppose for the time being that n_- is maintained constant by the addition of q anti-atoms per cm³ per second. Then $q \simeq 10^{-25} c n_+ n_-$, and if we use the above inequality for $n_+ n_-$, we find that $q \leqslant \sim 3\cdot10^{-22}$. The condition under which q would attain this upper limit would be if the interstellar clouds owed their turbulent motions to the pressure disturbances induced by the energy reservoir of fast electrons arising from annihilation. It is perhaps worth adding that the origin of the random velocities of the interstellar medium is not at present adequately explained, but we do not wish to stress this point. It may also be noted that the upper limit for q does not depend on a precise value of the annihilation cross-section, but it does depend on the lifetime of the electrons. This can be seen from an alternative derivation.

Under the conditions described above, a steady state is established, firstly between the addition of anti-atoms and their annihilation, and secondly between the production of electrons and their energy degradation. It follows then that an overall steady state must exist between the addition of anti-atoms and the degradation of the electrons, the precise value of the annihilation cross-section not entering into the latter balance at all. The rate at which energy is added to the reservoir of fast electrons is $\sim 3\cdot10^{-4}q$ erg per cm³

(3) L. H. MEREDITH, J. A. VAN ALLEN and M. B. GOTTLIEB: *Phys. Rev.*, **99**, 198 (1955).

(4) H. V. NEHER, V. Z. PETERSON and E. A. STERN: *Phys. Rev.*, **90**, 655 (1953).

per second. This input continuing over a period $\sim 10^{14}$ s must build up an energy density in the reservoir of about $3 \cdot 10^{10} q$ erg per cm³. If we set an upper limit of 10^{-11} erg per cm³ for this energy density then $q \leqslant \sim 3 \cdot 10^{-22}$.

2. – Applications to Cosmic Radio Sources.

In the remainder of this article we propose to consider the electrons derived from $\pm$ annihilation as possible sources of radio-waves, and we shall do this on the hypothesis that the upper limit for q is attained, i.e. $q = 3 \cdot 10^{-22}$. It is emphasized that this does not preclude other processes whereby high speed electrons might be produced such as have been proposed by one of us (5). No numerical estimates from now on depend on the precise value of the annihilation cross-section.

We shall consider, first emission within our own Galaxy, and second emission by extragalactic systems. Synchrotron radiation by 100 MeV electrons in a galactic magnetic field of 10^{-5} gauss has a critical frequency of only 2 MHz, which is far outside the observable radio band. Moreover radiation at this frequency would be strongly absorbed by the interstellar gas. The question accordingly arises as to what exceptional conditions are required in the Galaxy if observable radiation is to be emitted. Two possibilities suggest themselves:

(1) The electron energies are increased to values of the order of 1 GeV or more by a Fermi acceleration mechanism (2). The requirement that this mechanism overcomes the energy losses by atomic processes and by synchrotron emission at the threshold energy of the electrons is satisfied only if the turbulent velocities are very much greater than normal velocities, speeds of 10^2–10^3 km/s being required.

(2) In particular localities the magnetic field is as high as 10^{-4}–10^{-3} gauss. With field strengths of this order the critical frequency of a 100 MeV electron is raised into the observable radio-band, to 20–200 MHz.

These two possibilities are not of course mutually exclusive. Observable emission could arise from a combination of (1) and (2).

If (1) were operative the electrons would be expected to follow a cosmic ray type spectrum: i.e. the number of electrons with energies between E and $E+\mathrm{d}E$ proportional to $\mathrm{d}E/E^n$, n perhaps ~ 3. This leads to a radio spectrum $\sim \mathrm{d}\nu/\nu^{(n-1)/2}$. The spectrum in case (2), on the other hand, may be expected to be approximately proportional to $\mathrm{d}\nu$, much flatter than the form arising from (1). This difference suggests that we associate (1) with the strong radio

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(5) G. R. Burbidge: *Astrophys. Journ.*, **124** (September 1956) and *Phys. Rev.*, **103**, 264, (1956).

source in Cassiopeia which has a spectrum $\sim d\nu/\nu^{1.3}$, and that we associate (2) with the Crab Nebula which has an extremely flat spectrum [6]. Further discussion will be confined to the Crab, since the observational data for the Cassiopeia source remain incomplete.

In the Crab, analysis of the data on the visible radiation suggests that the mean magnetic field in the expanding nebulosity must be at least 10^{-3} gauss [7]. The likely presence of synchrotron radiation in the visible frequencies demands electron energies of the order of 10^{11} eV, so that Fermi acceleration processes and also perhaps a flux of high energy protons are demanded to produce these very high energies. It is probable that both the energy of the field and of the very high energy particles have been derived from the energy released in the initial supernova outburst. The point that we wish to make here is that the electrons responsible for the radio emission may come directly from annihilation processes. A small proportion of the annihilation electrons can also serve as injection electrons to produce much higher energies by Fermi processes.

The radius of the Crab is near to 1 psc [8] so that the volume swept out by the expanding nebulosity is now about 10^{56} cm^3. Using our upper limit of 10^{-11} erg per cm^3, the total energy content of the annihilation electrons that have become caught up in the nebulosity is about 10^{45} erg or about 10^{49} electrons in all. Now since the magnetic field is an agency whereby energy can be interchanged in some degree between the electrons, a trend towards the setting up of a Maxwell-Boltzmann distribution with a « kT » of about 100 MeV must arise. If we suppose that such a distribution represents a good approximation over the energy range of interest, the number of electrons with energy between E and $E+dE$ (E not too large compared with 100 MeV) is

$$\frac{2\cdot 10^{46}}{\pi} E^{\frac{1}{2}} \exp[-10^{-2}E]\, dE.$$

In this expression E is measured in MeV and the constant factor has been adjusted to give a total of 10^{49} electrons. Using this expression together with the well-known equation for the total energy loss of an electron by synchrotron emission, and carrying out an integration with respect to E, we find that the total radio energy output of the Crab is of the order of 10^{33} erg per second. This is in very satisfactory agreement with the estimates made from observational data, which give $\sim 7\cdot 10^{32}$ erg per second [9]. Furthermore the radio

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(6) J. L. PAWSEY: *Astrophys. Journ.*, **121**, 1 (1955).
(7) J. H. OORT and T. WALRAVAN: *Bull. Astron. Soc. Netherlands*, **12**, 285 (1956).
(8) W. BAADE: *Astrophys. Journ.*, **96**, 188 (1942).
(9) J. L. GREENSTEIN and R. MINKOWSKI: *Astrophys. Journ.*, **119**, 238 (1954).

spectrum obtained from this electron energy distribution is extremely flat, the relative power levels at 60, 160, and 3 200 MHz vary by less than 50 percent.

This is the first of a number of close numerical coincidences. A second coincidence arises when we consider the emission of radio waves by extragalactic systems. The radio energy emitted by extragalactic sources range from about 10^{37} erg per second for normal galaxies like our-own, e.g. M 31, to $\sim 10^{41}$ erg per second for M 87, and to $\sim 10^{44}$ erg per second for the source in Cygnus. This wide range of emission seems readily explicable on the basis that in normal galaxies such as M 31 there are neither intense large scale magnetic fields ($10^{-4} \div 10^{-3}$ gauss) nor widespread rapid turbulent gas motions. Synchrotron radiation by the annihilation electrons is then weak, except possibly in exceptional very localised regions within such a galaxy, and the frequency of the emission falls below the observable radio band. In the strong sources, on the other hand, it seems likely that the magnetic intensities and the turbulent velocities are both abnormally high.

Suppose we still take $q \cong 3 \cdot 10^{-22}$. Then the electron energy made available is $\sim 10^{-25}$ erg per cm^3 per second. The volume of M 87 is $\sim 10^{67}$ cm^3, so that the total electron energy made available in this galaxy is $\sim 10^{42}$ erg per second. The volume of the Cygnus source seems to be $\sim 10^{69}$ cm^3, giving an electron source $\sim 10^{44}$ erg per second, still for $q \cong 3 \cdot 10^{-22}$. Although these estimates are of necessity orders of magnitude, the coincidences with the observed rates of emission of radio waves give pause for reflection.

The question arises as to how exceptional galaxies may come to develop the special conditions necessary to become strong radio emitters. One speculation almost bordering on fantasy may be mentioned. There is the possibility that the Cygnus radio source consists of a galaxy in interaction with an anti-galaxy, or with a cloud of anti-gas. The annihilation process in such a case would continue until the total electron energy released became of the same order as the kinetic energy of collision, about 10^{60} erg. Such a situation would demand the annihilation of $\sim 10^{40}$ g, which is only about one part in 10^4 of the mass of a typical galaxy. At this stage the pressure exerted by the electrons via a magnetic field and possibly also via atomic collision processes would be sufficient to explode the colliding masses of gas, thereby slowing down and stopping the annihilation process.

It remains to add a few remarks concerning the cosmological implications of anti-matter. If as seems possible, the mean density of hydrogen in intergalactic space $\sim 6 \cdot 10^{-29}$ g per cm^3, the lifetime of an anti-atom against annihilation $\sim 10^{12}$ years. This time is so much greater than the inverse of Hubble's constant, $H^{-1} = 5.4 \cdot 10^9$ years, that the presence of anti-matter on a large scale cannot be ruled out as a cosmological speculation. This view is supported by the consideration that even the electrons resulting from annihilation would probably remain unobservable under the conditions oc-

would be necessary, however, to show that no annihilation disaster occurred curring in the intergalactic medium. In any development of $\pm$ cosmology it at earlier epochs, e.g. in the superdense initial state postulated in evolutionary cosmology.

Mention must finally be made of how the value of q might be maintained in our Galaxy and in other galaxies. We could suppose that — in our Galaxy (for instance) is being replenished by capture from the intergalactic medium, or we might accept a modified form of the steady-state theory of the expanding universe. In such a modified formulation $\pm$ would be regarded as originating in space at equal rates. The origin of would then take place at half the total rate that has been calculated for steady-state cosmology ([10]). With the currently accepted value of H, this gives $\sim 5\cdot 10^{-46}$ g per second for the rate of origin of -, and this yields a value of q in close agreement with the value $3\cdot 10^{-22}$ estimated above from widely different considerations.

It is emphasized that the present considerations are highly tentative. Yet it is difficult to avoid being impressed by the way in which such widely different issues as (i) the energy of agitation of the interstellar clouds in our own Galaxy, (ii) the radio emission by the Crab Nebula, (iii) the radio emission by M 87 and the Cygnus source, (iv) the rate of origin of matter in steady-state cosmology, become linked together by these considerations. It does not seem to us very likely that the numerical coincidences we have obtained could all be due to chance. This view can moreover be supported by a strong qualitative argument. In recent years it has become more and more clear that the radio waves from cosmic radio sources are derived from synchrotron emission, by high speed electrons. Evidently some widespread process for producing high-speed electrons is therefore required. Annihilation provides just such a process.

([10]) F. Hoyle: *Monthly Notices, Royal Astr. Soc.*, **108**, 372 (1948).

RIASSUNTO (*)

Si discutono le conseguenze dell'esistenza su scala cosmica di antimateria (sotto forma di positoni e antiprotoni). Con considerazioni sull'energia cinetica e magnetica delle nubi di gas interstellari e sulla energia della radiazione cosmica si dimostra che nella nostra Galassia il rapporto dell'antimateria alla materia ordinaria non può superare $\sim 10^{-7}$. Si esamina l'importanza che per la radioastronomia presenta il raggiungimento di tale limite. Sembra che questa ipotesi possa dare quantitativamente ragione delle intensità delle emissioni radio da sorgenti galattiche ed extra-galattiche. Si discute brevemente la relazione di queste idee con la cosmologia.

(*) *Traduzione a cura della Redazione.*

On the Existence of a Bound Nucleon-Λ^0 State.

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(ricevuto l'11 Giugno 1956)

Summary. — A variational method is used to find an upper limit to the strength of the nucleon-Λ^0 force from the measured binding energy of the hyper-triton. The force so found is too weak to lead to a bound nucleon-Λ^0 state.

While several nuclear emulsion events point unambiguously to the existence of a weakly bound neutron-proton-Λ^0 system, the « hyper-triton » (1), evidence for a bound proton-Λ^0 system is more uncertain (2). In fact recent arguments of Blatt and Butler (3) suggest that the nucleon-Λ^0 interaction is too weak to bind the two particles.

Using a variational method we have found an upper limit to the strength of the nucleon-Λ^0 force needed to bind the hyper-triton with a binding energy

(*a*) equal to B_D, the deuteron binding energy,

(*b*) 1 MeV in excess of B_D.

(*) Also supported by the Nuclear Research Foundation within the University of Sydney.

(1) A bibliography of all hyper-triton events to date is given by D. M. Haskin, T. Bowen, R. G. Glasser and M. Schein: *Phys. Rev.*, **102**, 244 (1956).

(2) G. Alexander, C. Ballario, R. Bizzarri, B. Brunelli, A. De Marco, A. Michelini, G. C. Moneti, A. Zavattini, A. Zichichi and J. P. Astbury: *Nuovo Cimento*, **2**, 365 (1955); F. Anderson, G. Lawlor and T. E. Nevin: *Nuovo Cimento*, **2**, 605 (1955); E. P. George, A. J. Herz, J. H. Noon and N. Solntseff: *Nuovo Cimento*, **3**, 94 (1956).

(3) J. M. Blatt and S. T. Butler: *Nuovo Cimento*, **3**, 409 (1956).

Let $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ denote respectively the position vectors of the two nucleons and the Λ^0 particle. On separating out the motion of the centre of mass, Schrödinger's equation becomes

$$\left[-\frac{\hbar^2}{M}\nabla_\varrho^2-\frac{\hbar^2}{2\mu}\nabla_r^2+V_{12}+V_{23}+V_{13}-E\right]\psi=0\,, \tag{1}$$

where $\boldsymbol{\nabla}_\varrho$ and $\boldsymbol{\nabla}_r$ denote the gradients with respect to the relative co-ordinates

$$\boldsymbol{\rho}=\boldsymbol{r}_2-\boldsymbol{r}_1\,,$$

$$\boldsymbol{r}=\boldsymbol{r}_3-\tfrac{1}{2}(\boldsymbol{r}_1+\boldsymbol{r}_2)\,.$$

In the equation $\mu=2MM_3/(2M+M_3)$ where M and M_3 are the nucleon and the Λ^0 masses respectively.

The following assumptions were made about the potentials:

(i) The neutron-proton interaction V_{12} is the same as in the deuteron (triplet interaction).

(ii) The proton-Λ^0 and the neutron-Λ^0 potentials V_{13} and V_{23} are equal, and both of the Yukawa form

$$-V_0\,\frac{\exp[-x/\beta]}{x/\beta}\,,$$

for a nucleon-Λ^0 separation x. For the moment we take V_{13} and V_{23} as spin-independent, and ignore the dependence of the wave function on the spin co-ordinate of the Λ^0 particle. However this restriction to spin-independent nucleon-Λ^0 forces is relaxed later.

We must consider what values of the « range » β of the nucleon-Λ^0 force are of physical interest. If a nucleon and a Λ^0 interact by exchanging a K-particle, then we expect a range β of the order $\hbar/m_K c$, where m_K is the K-meson mass. This estimate is about $\frac{1}{3}$ of the range of ordinary nuclear forces. However, such a short range is unlikely, since it is little more than the size of the supposed « hard core ». In the present paper we consider mainly values of β corresponding to ranges of the nucleon-Λ^0 potential between $\frac{1}{2}$ and 3 times that of internucleon forces.

With the assumed potentials the variation principle yields

$$V_0\int\left[\frac{\exp[-r_{13}/\beta]}{r_{13}/\beta}+\frac{\exp[-r_{23}/\beta]}{r_{23}/\beta}\right]\varphi^\dagger\varphi\,\mathrm{d}^3\boldsymbol{\rho}\,\mathrm{d}^3\boldsymbol{r}\leqslant \tag{2}$$
$$\leqslant\int\varphi^\dagger\left[-\frac{\hbar^2}{M}\nabla_\varrho^2+V_{12}+B_{\mathrm{D}}\right]\varphi\,\mathrm{d}^3\boldsymbol{\rho}\,\mathrm{d}^3\boldsymbol{r}-\int\varphi^\dagger\left[\frac{-\hbar^2}{2\mu}\nabla_r^2+\chi\right]\varphi\,\mathrm{d}^3\boldsymbol{\rho}\,\mathrm{d}^3\boldsymbol{r}\,.$$

for arbitrary, well-behaved functions φ, where

$$r_{13} = |\boldsymbol{r}_1 - \boldsymbol{r}_3|, \qquad r_{23} = |\boldsymbol{r}_2 - \boldsymbol{r}_3|$$

and $\chi = -E - B_D$, the « Λ^0 binding energy ».

As trial wave function we took

$$\varphi = \frac{\exp[-\xi r] - \exp[-\zeta r]}{r} \cdot \psi_D(\boldsymbol{\rho}) . \tag{3}$$

Here $\psi_D(\boldsymbol{\rho})$ is the deuteron wave function,

$$r = |\boldsymbol{r}_3 - \tfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)|, \qquad \frac{\hbar^2 \xi^2}{2\mu} = \chi ,$$

and ζ is a variation parameter ($\zeta > \xi$). Such a wave function has the right asymptotic form for large r. The product form (3) is a good approximation in the region $r \gg |\boldsymbol{\rho}|$ where the near equality of r_{13}, r_{23} and r makes the wave equation (1) approximately separable. The main contribution to the integrals in (2) comes from the region

$$\varrho \lesssim \varrho_D , \qquad r_{13} \sim r_{23} \sim \beta ,$$

where ϱ_D is the « radius of the deuteron », $\hbar/\sqrt{MB_D}$, and $\varrho = |\boldsymbol{\rho}|$. Only if $\beta \gg \varrho_D$ will this region coincide with the « good » region $r \gg \varrho$ of the trial wave function. Hence the wave function (3) will give accurate values of V_0 only for long range potentials V_{13} and V_{23}. However, the upper limit to V_0 given by the variation principle is sufficient to establish our result for all ranges β of physical interest.

To evaluate the integrals in (2) it is not necessary to know $\psi_D(\boldsymbol{\rho})$ explicitly. The first integral on the right is identically zero, while the other two integrals involve only the combination $[u^2(\varrho) + w^2(\varrho)]$, where we have written

$$\psi_D(\boldsymbol{\rho}) = \frac{1}{\varrho}\left[u(\varrho) + \frac{1}{\sqrt{8}}\, w(\varrho) S_{12}\right] \chi(1, 2).$$

S_{12} is the tensor operator and $\chi(1, 2)$ the triplet spin function. In our calculation we took

$$u^2(\varrho) + w^2(\varrho) = (\exp[-\delta\varrho] - \exp[-\gamma\varrho])^2 ,$$

where $\hbar^2\gamma^2/M = B_D$.

This choice of γ gives the correct asymptotic form for large ϱ. The constant δ

is found from the triplet effective range r_{ot}

$$r_{ot} = 2\int_0^\infty [\exp[-2\gamma\varrho] - [u^2(\varrho) + w^2(\varrho)]]\, d\varrho \, .$$

The table below gives the results of the variational calculation. b is the intrinsic range ([4]) of the nucleon-Λ^0 potential, i.e. $b = 2.1196\beta$. Rum on a is the intrinsic range of the Hulthen potential for which $(e^{-\gamma\varrho} - e^{-\delta\varrho})/\varrho$ is the exact deuteron space wave function, i.e., $a = 3/(\delta - \gamma) = 2.55 \cdot 10^{-13}$ cm. a serves merely as a unit of length. In the third column are the variational upper limits to V_0. The « well-depth parameters » ([4]) in the fourth column are found by dividing the values of V_0 from the third column by the minimum value of V_0 which would bind a nucleon to a Λ^0 particle. *A well-depth parameter less than unity therefore means a bound nucleon-Λ^0 state does not occur.*

Λ^0 binding energy	b/a	V_0 MeV	well-depth parameter
0	$\frac{1}{3}$	< 373	$< .94$
	$\frac{1}{2}$	< 128	$< .72$
	1	< 23.1	$< .52$
	2	< 4.73	$< .43$
	3	< 1.97	$< .40$
	∞	16.12 / $(b/a)^2$	.3643
1 MeV	$\frac{1}{2}$	< 174	$< .98$
	1	< 34.9	$< .79$
	2	< 8.67	$< .78$
	3	< 4.20	$< .86$

It is seen that the well-depth parameters are less than 1 for all physically likely values of the range b of the nucleon-Λ^0 potential. This is true even for a Λ^0 binding energy of 1 MeV. According to HASKIN *et al.* ([1]) the weighted average for this binding energy from all hyper-triton events to date is 0.5 ± 0.3 MeV. Thus, with the assumed potentials, the low binding energy of the hyper-triton is inconsistent with the existence of a bound nucleon-Λ^0 state.

Our calculation does not permit us to draw this conclusion for very short ranges b. However, the rapid increase in the variational upper limit to the well-depth as b decreases is almost certainly due to the poorness of the wave

([4]) J. M. BLATT and J. D. JACKSON: *Phys. Rev.*, **76**, 18 (1949).

function (3) for small b. In fact THOMAS [5] showed that a non-zero well-depth parameter for an attractive force of zero range between two pairs of particles in a three particle system would lead to infinite binding energy. The true well-depth parameters must therefore approach zero for very small ranges b [6]. An improved wave function of the Thomas type would therefore extend our conclusion to smaller values of b/a. However, such short ranges are so unlikely physically that we did not deem it worth the effort.

Let us now relax the assumption of the spin-independence of V_{13} and V_{23}. Suppose the nucleon-Λ^0 potential for a separation x is $V_+(x)$ in the nucleon-Λ^0 spin state with parallel spins, and $V_-(x)$ in the antiparallel spin state. We must now replace (3) by a wave function dependent on the spin co-ordinate of the Λ^0. Let $Y_{J,M}(1, 2, 3)$ denote the three-particle spin eigenfunctions of the spin operators $(\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3)^2$, $(\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3)_z$ and $(\mathbf{s}_1 + \mathbf{s}_2)^2$ with eigenvalues $J(J+1)\hbar^2$, $M\hbar$ and $2\hbar^2$ respectively. In place of (3) let us take the trial wave function

$$\varphi = \frac{\exp[-\xi r] - \exp[-\zeta r]}{r} \cdot \frac{u(\varrho) + (1/\sqrt{8})w(\varrho)S_{12}}{\varrho} \cdot Y_{J,M}(1, 2, 3)\,.$$

Then it can be shown that

$$(4) \qquad \int \varphi^+ (V_{13} + V_{23}) \varphi \, \mathrm{d}^3\boldsymbol{\rho}\, \mathrm{d}^3\boldsymbol{r} =$$

$$= \int \frac{(\exp[-\xi r] - \exp[-\zeta r])^2}{r^2} \frac{(u^2(\varrho) + w^2(\varrho))}{\varrho^2} (V_{\mathrm{eff}}(r_{13}) + V_{\mathrm{eff}}(r_{23}))\, \mathrm{d}^3\boldsymbol{\rho}\, \mathrm{d}^3\boldsymbol{r} +$$

$$+ \varkappa \int \frac{(\exp[-\xi r] - \exp[-\zeta r])^2}{r^2} \frac{w^2(\varrho)}{\varrho^2} (V_+(r_{13}) - V_-(r_{13}) + V_+(r_{23}) - V_-(r_{23}))\, \mathrm{d}^3\boldsymbol{\rho}\, \mathrm{d}^3\boldsymbol{r}\,,$$

where

$$V_{\mathrm{eff}}(x) = \frac{(J - s_{\Lambda^0} + 1)(J + s_{\Lambda^0})}{2(2s_{\Lambda^0} + 1)} V_+(x) + \frac{(-J + s_{\Lambda^0} + 1)(J + s_{\Lambda^0} + 2)}{2(2s_{\Lambda^0} + 1)} V_-(x)\,,$$

and $\varkappa$ is a constant depending on J and the spin s_{Λ^0} of the Λ^0 particle. Since the deuteron is predominantly in an S state, the second integral on the right of (4), which contains $w^2(\varrho)$ in the integrand, will be much smaller than the first, which involves $u^2(\varrho) + w^2(\varrho)$. In what follows we assume the second integral may be neglected. If we take

$$V_{\mathrm{eff}}(x) = -V_0 \frac{\exp[-x/\beta]}{x/\beta}\,,$$

[5] L. H. THOMAS: *Phys. Rev.*, **47**, 903 (1935).

[6] This limiting result is still true when V_{13} and V_{23} contain tensor forces if « zero range » now refers to the central part of these potentials.

then the calculation of an upper limit to V_0 is exactly as before. However V_0 is now the depth of the effective potential V_{eff}, while the potential operative in the two-particle nucleon-Λ^0 system is V_+ or V_-, whichever is the more attractive. If, therefore, our calculation of the well-depth parameter is still to be valid we must choose J to make V_{eff} equal to the more attractive of V_+ and V_-. The choice $J = s_{\Lambda^0}+1$ makes $V_{\text{eff}} = V_+$, while that of $J=|s_{\Lambda^0}-1|$ gives $V_{\text{eff}} = V_-$ if $s_{\Lambda^0} > \frac{1}{2}$ and $\frac{3}{4}V_- + \frac{1}{4}V_+$, if $s_{\Lambda^0} = \frac{1}{2}$. If $s_{\Lambda^0} = \frac{1}{2}$ we cannot make $V_{\text{eff}} = V_-$ using spin functions $Y_{J,M}(1, 2, 3)$ triplet in the nucleons. *Hence spin dependence in the nucleon-Λ^0 potential leaves our conclusion unaltered unless $s_{\Lambda^0} = \frac{1}{2}$* (7) *and in addition the singlet nucleon-Λ^0 spin state is the more attractive.*

The main inadequacy of the present calculation is the neglect of the possibility of non-central nucleon-Λ^0 forces. However we have *not* neglected non-central forces in the neutron-proton interaction.

We therefore conclude: Provided our assumptions about the potentials are not too far from the truth, the low binding energy of the hyper-triton indicates a nucleon-Λ^0 force too weak to bind these two particles into a hyper-deuteron or a hyper-dineutron.

* * *

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(7) A spin $s_{\Lambda_0} > \frac{1}{2}$ is suggested by the angular correlation measurements reported by W. B. FOWLER, R. P. SHUTT, A. M. THORNDIKE and W. L. WHITTEMORE: *Phys. Rev.*, **93**, 861 (1954) and **98**, 121 (1955); and by J. BALLAM, A. L. HODSON, W. MARTIN, R. R. RAU, G. T. REYNOLDS and S. B. TREIMAN: *Phys. Rev.*, **97**, 245 (1955).

RIASSUNTO (*)

Si impiega un metodo variazionale per trovare un limite superiore per la forza del nucleone Λ^0 partendo dall'energia di legame dell'iper-tritone risultante dalle misure. La forza così trovata è troppo debole per condurre a uno stato legato del nucleone Λ^0.

(*) *Traduzione a cura della Redazione.*

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Contribution to Stephenson-Kilmister's Unified Theory of Gravitation and Elektromagnetism.

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(ricevuto il 13 Giugno 1956)

Summary. — A geometrical interpretation for Stephenson-Kilmister's unified theory of gravitation and electromagnetism is given on the base of Finslerian geometry. It is known that the correct equations of motion for a charged test body moving in the Riemannian space under the ponderomotorical influence of the electromagnetic field can be deduced from a variational principle with the Lagrangian: $\mathcal{F}(x, \mathrm{d}x) = A_\mu(x)\,\mathrm{d}x^\mu + \{g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\}^{\frac{1}{2}}$, where $A_\mu(x)$ and $g_{\mu\nu}(x)$ are the components of the potential of the electromagnetic field and the components of the metrical ground tensor of the original Riemannian space, respectively. Considering $\mathcal{F}(x, \mathrm{d}x)$ as the ground function of a Finsler's space a reformulation of Stephenson-Kilmister's idea is suggested.

In the first periode of its development the general theory of relativity was based on the so-called geodetic axiom according to which the geometrical structure of Riemannian space has to be determined so that the path of a test body moving inertially—i.e. moving under the ponderomotoric influence of gravitation alone—would be identical with one of the geodetic lines of the space. Recently this geodetic axiom, owing to the well known result of Einstein, Infeld and Hoffmann (1) and Einstein and Infeld (2) has lost its interest, namely, on the base of the mentioned results the equations of motion can be derived directly from the field equations.

In the case of the unified theory of gravitation and electromagnetism too investigations were at first naturally also based on the geodetic axiom. But

(1) A. Einstein, L. Infeld and G. Hoffmann: *Ann. Math.*, **39**, 66 (1938).

(2) A. Einstein and L. Infeld: *Can. Journ. of Math.*, **1**, 209 (1949).

after the results of EINSTEIN and his co-workers it seemed to be more suitable to look for a direct generalization of the general theory of relativity based on the Hamiltonian formalism. Undoubtedly, the most natural generalizations of the general theory of relativity are the affine field theories developed by EINSTEIN (3), by EINSTEIN and STRAUS (4) and by SCHRÖDINGER (5) respectively. However, these very interesting theories are only adequate from the mathematical point of view, namely in the case of both of the theories of Einstein, based on the investigation of INFELD (6), and in the case of Schrödinger's theory, based on the investigation of IKEDA (7) it is known that the correct equations of motion cannot by deduced from the field equations. It is very difficult to understand the reason of the inadequateness of these theories, but it is not impossible that it can be found in the fact of the lacking of a quasi-geometrical criterium in the case of these theories which were discussed in a previous paper (8).

As matters stand, some interest can again be attached to generalizations of the general theory of relativity based on the geodetic axiom.

It is known that in the Riemannian space the correct equations of motion for a charged test body with charge $e = 1$ are

$$\frac{d^2x^\lambda}{ds} + \Gamma_{\mu\cdot\nu}^{\ \lambda} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + g^{\lambda\varrho}F_{\nu\varrho}\frac{dx^\nu}{ds} = 0 \tag{1}$$

with

$$F_{\varrho\nu} = \partial_\nu A_\varrho - \partial_\varrho A_\nu \,. \tag{2}$$

where A_ϱ are the components of the potential of the electromagnetic field.

Now, the following problem can be formulated: *the Lagrangian*

$$d\sigma = \mathcal{F}(x, dx) \tag{3}$$

of the variational problem

$$\delta \int d\sigma = 0 \tag{4}$$

should be found so that the Euler-Lagrange equations of our problem (4) *would be just identical with the equations of motion* (1).

(3) A. EINSTEIN: *Ann. Math.*, **46**, 578 (1945); *The Meaning of Relativity* (London, 1950), Appendix 2.

(4) A. EINSTEIN and E. G. STRAUS: *Ann. Math.*, **47**, 731 (1946).

(5) E. SCHRÖDINGER: *Proc. Roy. Irish Acad.*, **51**, 163, 205 (1947); **52**, 1 (1948).

(6) L. INFELD: *Acta Phys. Polon.*, **10**, 284 (1951).

(7) M. IKEDA: *Prog. Theor. Phys.*, **7**, 127 (1952).

(8) J. I. HORVÁTH: *Bull. Acad. Pol.*, **3**, 151 (1955).

The solution of this problem solved recently by STEPHEHSON and KILMISTER (9) on the basis of another starting point, can be obtained as follows

$$d\sigma = A_\mu(x)\,dx^\mu + \{g_{\mu\nu}(x)\,dx^\mu\,dx^\nu\}^{\frac{1}{2}} \equiv \mathcal{F}(x, dx)\,. \tag{5}$$

Proof. Writing the equation (4) in the form

$$\delta \int \left\{ A_\mu \frac{dx^\mu}{dt} + \left[g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right]^{\frac{1}{2}} \right\} dt = 0\,,$$

the Euler-Lagrange equation of our variational problem using the usual parameter

$$ds \overset{\text{def}}{=} \{g_{\mu\nu}\,dx^\mu\,dx^\nu\}^{\frac{1}{2}}$$

of the original Riemannian geometry is given by

$$\partial_\varrho \mathcal{F}\left(x, \frac{dx}{ds}\right) - \frac{d}{ds} \frac{\partial \mathcal{F}(x, dx/ds)}{\partial(dx_\varrho/ds)} = 0\,,$$

or explicitly

$$(\partial_\varrho A_\mu)\dot{x}^\mu + \frac{1}{2}(\partial_\varrho g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu - \frac{dA_\varrho}{ds} - \frac{d}{ds}\{[g_{\varkappa\lambda}\dot{x}^\varkappa \dot{x}^\lambda]^{\frac{1}{2}} g_{\mu\varrho}\dot{x}^\mu\} = 0$$

with

$$\dot{x}^\mu = \frac{dx^\mu}{ds}\,,$$

taking into account that

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 1\,.$$

It can be obtained directly

$$(\partial_\varrho A_\mu)\dot{x}^\mu + \frac{1}{2}(\partial_\varrho g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu - (\partial_\mu A_\varrho)\dot{x}^\mu - (\partial_\tau g_{\mu\varrho})\dot{x}^\mu \dot{x}^\tau - g_{\mu\varrho}\frac{d^2 x^\mu}{ds^2} = 0$$

and

$$(\partial_\varrho A_\mu - \partial_\mu A_\varrho)\dot{x}^\mu - \frac{1}{2}(\partial_\nu g_{\mu\varrho} + \partial_\mu g_{\nu\varrho} - \partial_\varrho g_{\mu\nu})\dot{x}^\mu \dot{x}^\nu - g_{\mu\varrho}\frac{d^2 x^\mu}{ds^2} = 0$$

respectively, finally based on (2) we have

$$g_{\mu\varrho}\frac{d^2 x^\mu}{ds^2} + \Gamma_{\mu\varrho\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + F_{\mu\varrho}\frac{dx^\mu}{ds} = 0$$

which is equivalent with (1). *Q.e.d.*

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(9) G. STEPHENSON and C. W. KILMISTER: *Nuovo Cimento*, **10**, 230 (1953).

As a result of the above Stephenson and Kilmister's unified theory of gravitation and electromagnetism can be built up in the following way (9):

The equations (1) define the paths of test particles in terms of the field variables which must be determined by the field equations. Based on the well known method of general theory of relativity a variational principle must be employed to determine the field equations and to adopt as a Lagrangian density $R\sqrt{|g|}$, where R is the scalar curvature of space and g is the determinant of the $g_{\mu\nu}$. The geometrical definition of the scalar curvature of the space is as follows: a set of four perpendicular directions at a point determines six geodesic surfaces and the mean of their Gaussian curvatures being independent of the directions chosen is proportional to the scalar curvature R. On the basis of the metric introduced by (5) this mean Gaussian curvature was obtained by STEPHENSON and KILMISTER as

$$\frac{1}{12}\left\{R + \frac{1}{2} F_{\mu\nu}F^{\mu\nu}\right\}.$$

This result suggests the adoption of a Hamiltonian of the form

$$\mathcal{J} = \int \{R + \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu}\}\sqrt{|g|}\,\mathrm{d}^4x \tag{6}$$

leading to the well known field equations

$$(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) + (F_{\mu\varrho}F_{\mu}^{\cdot\varrho} - \tfrac{1}{4}g_{\mu\nu}F_{\varrho\sigma}F^{\varrho\sigma}) = 0\,, \tag{7}$$

$$\frac{1}{\sqrt{|g|}}\,\partial_\nu(F^{\mu\nu}\sqrt{|g|}) = 0\,. \tag{8}$$

Equations (7) are Einstein's field equations of the gravitational field, whilst equations (8) correspond to one set of Maxwell's equations. The problem of the equations of motion (1) implied by these equations has been considered by INFELD and WALLACE [10] on the basis of the known limiting process.

Therefore, it is possible to accept the generalization of the Riemannian geometry in which the infinitesimal distance of two neighbouring points is given by $\mathrm{d}\sigma$ as a geometrical basis of a unified theory of first kind [11], but from the geometrical point of view this generalization of Riemannian geometry is not a natural one. Namely, in this generalization of the Riemannian geometry $\mathrm{d}\sigma^2$ is not the usual quadratic form of $\mathrm{d}x^\mu$ occurring in geometry.

[10] L. INFELD and P. R. WALLACE: *Phys. Rev.*, **57**, 797 (1940).
[11] J. I. HORVÁTH and A. MOÓR: *Zeits. f. Phys.*, **131**, 544 (1952).

However, if we regard $\mathcal{F}(x, \mathrm{d}x)$ as a ground function of a Finsler space [12], we can immediately find a quite natural generalization of the Riemannian space as a geometrical basis for an unified theory of fields.

The metrical ground tensor of our Finsler space with the ground function $\mathcal{F}(x, \mathrm{d}x)$ can be calculated easily [10] and it is given by

$$\begin{aligned}(9)\qquad \gamma_{\mu\nu}(x, \dot{x}) = {} & A_{\mu}(x)A_{\nu}(x) + g_{\mu\nu}(x) + G^{-1}\{A_{\mu}(x)g_{\nu\varrho}(x) + A_{\nu}(x)g_{\mu\varrho}(x)\}\dot{x}^{\varrho} + \\ & - G^{-1}A_{\varrho}(x)\dot{x}^{\varrho}g_{\mu\nu}(x) + G^{-3}A_{\varrho}(x)g_{\mu\sigma}(x)g_{\nu\tau}(x)\dot{x}^{\varrho}\dot{x}^{\sigma}\dot{x}^{\tau}\,,\end{aligned}$$

with

$$G = \{g_{\varrho\sigma}(x)\dot{x}^{\varrho}\dot{x}^{\sigma}\}^{\frac{1}{2}}\,.$$

The tensor which determines the torsion of the space is given by

$$\begin{aligned}(10)\qquad C_{\lambda\mu\nu} = {} & \tfrac{1}{2}\{G^{-1}(A_{\lambda}g_{\mu\nu} + A_{\mu}g_{\nu\lambda} + A_{\nu}g_{\lambda\mu}) - \\ & - G^{-3}(A_{\lambda}g_{\mu\varrho}g_{\nu\sigma} + A_{\mu}g_{\nu\varrho}g_{\lambda\sigma} + A_{\nu}g_{\lambda\varrho}g_{\mu\sigma})\dot{x}^{\varrho}\dot{x}^{\sigma} - \\ & - A_{\lambda}\dot{x}^{\lambda}[G^{-3}(g_{\lambda\mu}g_{\nu\varrho} + g_{\mu\nu}g_{\lambda\varrho} + g_{\nu\lambda}g_{\mu\varrho})\dot{x}^{\varrho} + 3G^{-5}g_{\lambda\varrho}g_{\mu\sigma}g_{\nu\tau}\dot{x}^{\varrho}\dot{x}^{\sigma}\dot{x}^{\tau}]\}\,.\end{aligned}$$

The equation of the geodetic lines of the space can be given in the following form

$$(11)\qquad \frac{\mathrm{d}^2x^{\lambda}}{\mathrm{d}\sigma^2} + \Gamma_{\varrho\cdot\tau}^{\ \lambda}\frac{\mathrm{d}x^{\varrho}}{\mathrm{d}\sigma}\frac{\mathrm{d}x^{\tau}}{\mathrm{d}\sigma} = GF^{\lambda}_{\cdot\varrho}\frac{\mathrm{d}x^{\varrho}}{\mathrm{d}\sigma} + K^{\lambda}$$

with

$$\Gamma_{\mu\cdot\nu}^{\ \lambda} = g^{\lambda\varrho}\Gamma_{\mu\varrho\nu}\,; \qquad \Gamma_{\mu\varrho\nu} = \tfrac{1}{2}\{\partial_{\mu}g_{\varrho\nu} + \partial_{\nu}g_{\mu\varrho} - \partial_{\varrho}g_{\mu\nu}\}$$

$$\sigma = \int_{t_0}^{t}\mathcal{F}\left(x, \frac{\mathrm{d}x}{\mathrm{d}\tau}\right)\mathrm{d}\tau$$

$$F^{\lambda}_{\cdot\varrho} = g^{\lambda\mu}F_{\mu\varrho}\,, \qquad K^{\lambda} = 2g^{\lambda\varkappa}G_{\varkappa}\,,$$

$$\begin{aligned}G_{\varkappa} = {} & \tfrac{1}{2}A_{\varrho}\dot{x}^{\varrho}F_{\tau\varkappa}\dot{x}^{\tau} - \tfrac{1}{2}A_{\varrho}\dot{x}^{\varrho}G^{-1}(\partial_{\sigma}g_{\varkappa\tau} - \tfrac{1}{2}\partial_{\varkappa}g_{\sigma\tau})\dot{x}^{\tau}\dot{x}^{\sigma} - \\ & - \tfrac{1}{4}G^{-1}[A_{\varkappa}(\partial_{\varrho}g_{\sigma\tau})\dot{x}^{\sigma}\dot{x}^{\tau} + 2(\partial_{\varrho}A_{\sigma})g_{\varkappa\tau}\dot{x}^{\sigma}\dot{x}^{\tau} - G^{-2}A_{\lambda}\dot{x}^{\lambda}g_{\varkappa\tau}(\partial_{\varrho}g_{\nu\sigma})\dot{x}^{\tau}\dot{x}^{\sigma}] + \tfrac{1}{2}A_{\varkappa}(\partial_{\varrho}A_{\sigma})\dot{x}^{\varrho}\dot{x}\,.\end{aligned}$$

Now, by the transformation of the parameter σ to

$$s = \int_{t_0}^{t}\left\{g_{\varrho\sigma}(x)\,\frac{\mathrm{d}x^{\varrho}}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}\tau}\right\}^{\frac{1}{2}}\mathrm{d}\tau\,,$$

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[12] J. I. Horváth: *Phys. Rev.*, **80**, 901 (1950).

or rather by direct calculation it can be proved that our equations (11) are identical with the equations of motion (1).

Undoubtedly, this geometrization of gravitation and electromagnetism is a very natural one, nevertheless, unfortunately, in this state of its development this theory lacks completeness. Namely, the geometrical structure of space is determined by the metrical ground tensor (9) and by the tensor of torsion (10), which depend on the tensors $g_{\mu\nu}(x)$ and $A_\mu(x)$ being—in the case of the empty space—the solutions of the simultaneous differential equations (7) and (8) respectively, which are the Euler-Lagrange equations of the Hamiltonian (6), the Lagrangian of which is given by

$$L = R + \tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} .$$

It has been shown above that this Lagrangian was the scalar curvature of Stephenson-Kilmister's generalized Riemannian space. Naturally, it seems very probable that either the L has a similar geometrical meaning in the Finslerian geometry too, or that, at least, it can be replaced by a suitable scalar of the geometry. However, the investigations in this direction have so far not produced the expected result.

RIASSUNTO (*)

Sulla base della geometria di Finsler si dà un'interpretazione geometrica della teoria unificata della gravitazione e dell'elettromagnetismo di Stephenson e Kilmister. Si sa che le corrette equazioni di moto per un corpo di prova carico moventesi nello spazio di Riemann sotto l'influenza ponderomotrice del campo elettromagnetico si può dedurre da un principio variazionale col lagrangiano: $\mathcal{F}(x, dx) - A_\mu(x)\, dx^\mu + \{g_{\mu\nu}(x)\, dx^\mu dx^\nu\}^{\frac{1}{2}}$, dove $A_\mu(x)$ e $g_{\mu\nu}(x)$ sono rispettivamente le componenti del potenziale del campo elettromagnetico e le componenti del tensore fondamentale dello spazio di Riemann originale. Considerando $\mathcal{F}(x, dx)$ come funzione fondamentale di uno spazio di Finsler si suggerisce una nuova formulazione della teoria di Stephenson e Kilmister.

(*) *Traduzione a cura della Redazione.*

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Contributions to the Unified Theory of Physical Fields.

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(ricevuto il 13 Giugno 1956)

Summary. — Discussing EINSTEIN's criteria of unified theory of physical fields it is suggested that the second criterium: « Neither the field equations nor the Hamiltonian function can be expressed as a sum of several invariant parts, but are formally unified entities » should be replaced by the weaker one: « *Neither the field equations nor the Hamiltonian function can be expressed as the sum of several invariant parts which have an individual geometrical meaning* » and EINSTEIN's criteria should be completed by the following: « *From the field equations of a (non-linear) unified field theory the correct equations of motion must be deduced* ».

Ten years ago A. EINSTEIN (1) formulated the general idea of the unified theory of physical fields according to which every attempt to establish a unified field theory must start from a transformation which is no less general than that of the continuous transformations of the space-time co-ordinates. It is further reasonable to attempt the establishment of a unified theory by a generalization of the relativistic theory of gravitation. This means that—according to EINSTEIN's view if we speak about a unified theory of physical fields we have two possible points of view, the distinction of which is essential for the following:

1) *That the field appears as a unified covariant entity.*

As an example, EINSTEIN has cited the unification of the electric and the magnetic fields by the special theory of relativity. The unification here consists of the entire field considered being described by a skew-symmetric tensor.

(1) A. EINSTEIN: *Ann. of Math.*, **46**, 578 (1945).

The basic group of Lorentz transformations does not enable us to split this field indepedently of the system of co-ordinates, into an electric and a magnetic one.

2) *Neither the field equations nor the Hamiltonian function can be expressed as a sum of several invariant parts, but are formally unified entities*

This criterion of uniformity is also satisfied in the example of the special relativistic description of Maxwell's equation and can be regarded as a special kind of irreducibility of the field theory.

In the course of the last ten years many researchers have dealt with the formation of unified field theories based on different geometries of the four-dimensional space-time world which fulfil the general criteria of EINSTEIN mentioned above. But these have shown that special difficulties arise at the realization of EINSTEIN's programme.

1. In the case of EINSTEIN's first criterion we have to consider that any experimental effect which can be regarded as an experimental basis for this criterion is unknown In the theory of relativistic electrodynamics the well known effect of Wien proves experimentally the unified covariance of the electric and magnetic field, but till now it is not known that such a direct interaction exists between the electromagnetic and gravitational fields.

Presumably the lack of experimental evidence of this interaction in the case of the electromagnetic and gravitational fields might be proved theoretically as follows:

According to the interpretation of Einstein's theory of gravitation introduced by N. ROSEN (2), A. PAPAPETRU (3), M. KOHLER (4) and S. N. GUPTA (5) respectively, this theory of Einstein can be regarded as a field theory of the second kind (6) in the pseudo-Euclidian space. In this, way the gravitational field is described by the ten gravitational potentials $g_{\mu\nu}$ in the same manner as the electromagnetic field is described by the field tensor $F^{\mu\nu}$. In the case of field theories of the second kind the interaction of two fields is defined by an invariant constructed by the field quantities of both fields. But the most natural term of interaction of electromagnetism and gravitation $g_{\mu\nu}F^{\mu\nu} = 0$; in other words the term of interactions vanishes.

It seems that this argument is in accordance with the experimental facts, but, of course, it is possible that there exists some other type of interaction

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(2) N. ROSEN: *Phys. Rev.*, **57**, 147 (1940).

(3) A. PAPAPETRU: *Proc. Roy. Irish Acad.*, **52**, 11 (1948).

(4) M. KOHLER: *Zeits. f. Phys.*, **131**, 57 (1952).

(5) S. N. GUPTA: *Proc. Phys. Soc. London*, A **64**, 426 (1951); **65**, 161, 608 (1952).

(6) J. I. HORVÁTH: *Bull. Acad. Pol.*, **3**, 151 (1955).

between the two fields. In this case the final unified field theory which fulfils Einstein's first criterion has to predict the experimental evidence of the interaction between the electromagnetic and gravitational fields.

2. The criterion of irreducibility of the field theory is mentioned by EINSTEIN as a weaker one. Despite of this view, some difficulties arose in the course of the last ten years in connection with this criterion too.

Undoubtedly the most natural generalizations of the general theory of relativity were the affine field theories developed by A. EINSTEIN (1), by A. EINSTEIN and E. G. STRAUSS (7), and by E. SCHRÖDINGER (8) respectively. However, these very interesting theories are only adequate from the mathematical point of view. Namely, in the case of both theories of Einstein on the basis of the investigation of L. INFELD (9), and in the case of Schrödinger's theory based on the investigation of M. IKEDA (10), it is known that the correct equations of motion cannot be deduced from the field equations.

Recently W. B. BONNOR (11) has proved that by changing the field equations of Einstein's non-symmetric unified theory of gravitation and electromagnetism it is possible to make them imply the existence of the Coulomb force between electric charges, and also that the equation of motion of charged masses follows correctly to the order of approximation considered. But it is interesting that to obtain this result BONNOR had to complete Einstein's Hamiltonian by an additive scalar to be able to derive the correct field equations.

This means that in the case of field theories which fulfil Einstein's second criterion it is apparently not possible to derive the equation of motion from the field equations, and if this deduction of the equations of motion is possible the theory does not fulfil the criterion of irreducibility.

3. We have mentioned above some arguments which indicate the difficulties in connection with the criteria of EINSTEIN. However, in the absence of experimetal effects which render possible an objective choice among the different types of field theories, the final criteria of the unified theories of physical fields may play a decisive role, whenever the acceptability of a considered field theory is examined. As matters stand there is urgent need to find some suitable citeria of the unified field theory for the selection of the acceptable unified theories, even if these criteria have only some importance at the present stage of development.

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(7) A. EINSTEIN and E. G. STRAUSS: *Ann. of Math.*, **47**, 731 (1946).
(8) E. SCHRÖDINGER: *Proc. Roy. Irish Acad.*, **51**, 163. 205 (1947); **52**. 1 (1948).
(9) L. INFELD: *Acta Phys. Pol.*, **10**, 284 (1951).
(10) M. IKEDA: *Prog. Theor. Phys.*, **7**, 127 (1952).
(11) W. B. BONNOR: *Proc. Roy. Soc. London*, A **226**, 366 (1954).

First of all based on the result of A. EINSTEIN and J. GROMMER (12)—in the case of a non-linear theory—the correct equation of motion must be deduced from the field equations. This is nowadays a generally accepted point of view and has the advantage that by means of the well known methods of approximation (13-16) on this basis, in the case of every unified field theory, the deducibility of the equations of motion from the field equation can be examined.

Under these circumstances we can consider as a final criterion of unified theories of physical fields that

3) *From the field equation of a (non-linear) unified field theory the correct equation of motion must be deduced.*

Naturally, the existential problem can arise whether our criterion 3) is compatible with criterion 2) of EINSTEIN, since recent results seem to prove their incompatibility. It is very difficult to solve this problem generally, but it is not doubtful that in the case of incompatibility of the two criteria, only 2) can be given up.

Now, an alternative formulation of Einstein's second criterion will be suggested as follows:

2′) *Neither the field equation nor the Hamiltonian function can be expressed as the sum of several invariant parts which have an individual geometrical meaning.*

This formulation is essentially weaker as Einstein's original one and is based on the following arguments:

In the case of general theory of relativity, which theory is the prototype of the field theories of the first kind it is not only characteristic for the Hamiltonian function that the latter is a formally unified entity, but it is very essential that it has a direct geometrical meaning; namely, the Hamiltonian function of the general theory of relativity is the scalar curvature of the space. It is possible to derive a unified field theory based on a Hamiltonian function, which fulfils our criterion 2′), but cannot be expressed as the sum of several invariant parts which have geometrical meaning separately, this theory seems to be compatible with the ideas of the unified field theories of the first kind.

By suitable generalization of the metric of Riemannian space (17,18) the

(12) A. EINSTEIN and J. GRONNER: *Berl. Ber. Akad.*, 2 (1927).
(13) A. EINSTEIN, L. INFELD and B. HOFFMANN: *Ann. of Math.*, **39**, 65 (1938).
(14) A. EINSTEIN and L. INFELD: *Can. Journ. Math.*, **1**, 269 (1949).
(15) V. A. FOCK: *Journ. Phys. USSR*, **1**, 81 (1939).
(16) A. PAPAPETRU: *Fotrschr. d. Phys.*, **1**, 29 (1953).
(17) G. STEPHENSON and C. W. KILMISTER: *Nuovo Cimento*, **10**, 230 (1953).
(18) J. I. HORVÁTH: *Nuovo Cimento*, **4**, 571 (1956).

scalar curvature of the space can be expressed as follows:

$$\frac{1}{12}\left\{R + \tfrac{1}{2}F_{\varrho\sigma}F^{\varrho\sigma}\right\},$$

where R is the scalar curvature of the original Riemannian space wih the metrical ground tensor $g_{\mu\nu}(x)$ and $F_{\mu\nu}(x)$ is the field tensor of the electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(the A_μ's are the potentials of the electromagnetic field). By independent variations of A_μ and $g_{\mu\nu}$, the field equations and—based on the known methods of approximation—from the field equations, the correct equations of motion can be deduced. This theory sketched in broad lines, should only prove the satisfiableness and general idea of our criterion 2′).

RIASSUNTO (*)

Discutendo i criteri di EINSTEIN per una teoria unificata dei campi, l'autore suggerisce di sostituire al secondo criterio: « Nè le equazioni di campo nè l'hamiltoniana si possono esprimere come somma di invarianti, ma sono entità formalmente unificate », il più debole enunciato: « *Nè le equazioni di campo nè l'hamiltoniana si possono esprimere come somma di invarianti aventi individualmente un significato geometrico* » e i criteri di EINSTEIN dovrebbero essere completati dal seguente: « *Dalle equazioni di campo di una teoria unificata (non lineare) debbono potersi dedurre le corrette equazioni di moto* ».

(*) *Traduzione a cura della Redazione.*

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On a New Approximate One-velocity Theory of Multiple Scattering in Infinite Homogeneous Media.

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Summary. — Within the framework of a new approximate formalism for solving one-velocity multiple scattering problems in infinite homogeneous media, which was presented in an earlier paper, the appropriate expressions for the particle current are derived and carefully discussed in both cases of isotropic and slightly non-isotropic scattering. A detailed comparison between the corresponding rigorous and approximate formulae is carried out in the special case of an isotropic point source and leads to the conclusion that the newly derived approximations are highly accurate and possess remarkable properties.

1. – Introduction.

The calculations presented in this article can in fact be regarded as a complement to our previous paper [1], in which a high accuracy approximation for solving multiple scattering problems in infinite homogeneous media has been obtained in both cases of isotropic and slightly non-isotropic scattering. A peculiar aspect in the development of our formalism is that we did not need to know the appropriate general expressions for the particle current $\boldsymbol{j}(\boldsymbol{r})$ corresponding with the density $\varrho(\boldsymbol{r})$ within the framework of the theory. However, if we wish to extend the formalism to finite medium problems, it becomes necessary to introduce the current, because such problems require the formulation of boundary conditions in which $\boldsymbol{j}$ plays an important role. There-

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(1) C. C. Grosjean: *Nuovo Cimento*, **3**, 1262 (1956).

fore, the present article concerns the derivation of the general approximate expressions for the current vector in both cases of isotropic and slightly non-isotropic scattering, and presents a detailed discussion on the validity of the obtained formulae. We keep essentially the same symbols as those used in ref. [1].

2. – Theoretical Development.

2·1. *Calculation of the current vector.* – Let $S(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$ again describe an arbitrary given distribution of isotropic sources present in an infinite homogeneous medium. Let us first consider the case of isotropic scattering. The derivation of $\boldsymbol{j}$ can be nicely achieved on the basis of the four rigorous relations mentioned by RICHARDS in his paper [2], namely (in our notations):

$$\nabla\cdot\boldsymbol{j} = S(\boldsymbol{r}) - (1-\omega)v\varrho(\boldsymbol{r})/\lambda\,, \tag{1}$$

$$\nabla\cdot\boldsymbol{j}_d = S(\boldsymbol{r}) - v\varrho_d(\boldsymbol{r})/\lambda\,, \tag{2}$$

$$\nabla\times\boldsymbol{j} = 0\,, \qquad \nabla\times\boldsymbol{j}_d = 0\,, \tag{3}$$

in which the quantities ϱ_d and $\boldsymbol{j}_d$ again refer to « direct beam » contributions from the source distribution. Eqs. (3) tell us that both $\boldsymbol{j}$ and $\boldsymbol{j}_d$ can be represented as gradients of scalar functions, whereas eqs. (1) and (2) are essentially continuity equations. Multiplying (2) with $(1-\omega)$ and subtracting it from (1), we find:

$$\begin{aligned}\nabla\cdot[\boldsymbol{j}-(1-\omega)\boldsymbol{j}_d] &= \omega S(\boldsymbol{r}) - (1-\omega)v[\varrho(\boldsymbol{r})-\varrho_d(\boldsymbol{r})]/\lambda = \\ &= \omega S(\boldsymbol{r}) - (1-\omega)v\varrho_s(\boldsymbol{r})/\lambda\,.\end{aligned} \tag{4}$$

Now, let us introduce our approximation for the density of scattered particles ϱ_s. Multiplying (12) of ref. [1] by $(2-\omega)\lambda v/3$, our basic equation for ϱ_s becomes:

$$\frac{(2-\omega)\lambda v}{3}\nabla^2\varrho_s(\boldsymbol{r}) - \frac{(1-\omega)v}{\lambda}\varrho_s(\boldsymbol{r}) + \omega S(\boldsymbol{r}) = 0\,, \tag{5}$$

which, combined with (4), leads to the result

$$\nabla\cdot[\boldsymbol{j}-(1-\omega)\boldsymbol{j}_d] = -\frac{(2-\omega)\lambda v}{3}\nabla^2\varrho_s\,.$$

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[2] P. I. RICHARDS: *Phys. Rev.*, **100**, 517 (1955).

or

$$\nabla \cdot \left[\boldsymbol{j} - (1-\omega)\boldsymbol{j}_d + \frac{(2-\omega)\lambda v}{3} \nabla \varrho_s \right] = 0 \, . \tag{6}$$

This approximate equation is so similar to Richards' eq. (12), that on the basis of the same reasoning as the one presented by this author, we can be assured that the vector between the brackets vanishes. Consequently, we find as general expression for the current vector in the case of isotropic scattering:

$$\boxed{\boldsymbol{j} \simeq (1-\omega)\boldsymbol{j}_d - \frac{(2-\omega)\lambda v}{3} \nabla \varrho_s} \, , \tag{7}$$

in which the sign $\simeq$ symbolizes the fact that the right hand side is an approximation, which is of course the correct one within the new formalism. Here, exactly like any « direct beam » contribution appearing in the theory, $\boldsymbol{j}_d$ represents the rigorous expression for the direct current vector, namely:

$$\boldsymbol{j}_d = \frac{1}{4\pi} \iiint_{\infty}^{\infty} S(\boldsymbol{r}') \, \frac{(\boldsymbol{r}-\boldsymbol{r}') \exp[-|\boldsymbol{r}-\boldsymbol{r}'|/\lambda]}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \, \mathrm{d}\boldsymbol{r}' \, . \tag{8}$$

In the case of pure capture ($\omega = 0$), there are no scattered particles ($\varrho_s = 0$) (cfr. ref. (1), eq. (10)) and (7) reduces to the exact result

$$\boldsymbol{j} = \boldsymbol{j}_d \, . \tag{9}$$

In the case of pure diffusion ($\omega = 1$), our formula yields

$$\boldsymbol{j} = -\frac{\lambda v}{3} \nabla \varrho_s \, , \tag{10}$$

which differs only from the elementary diffusion theory approximation by the fact that ϱ_s replaces the total density ϱ.

For any value of ω, $\boldsymbol{j}$ can be correctly separated in two parts: the rigorous « direct beam » current $\boldsymbol{j}_d$ and the approximate diffuse current $\boldsymbol{j}_s$ given by

$$\boldsymbol{j}_s = -\omega \boldsymbol{j}_d - \frac{(2-\omega)\lambda v}{3} \nabla \varrho_s \, , \tag{11}$$

which surprisingly contains a negative fraction of the « direct beam » contribution (except when $\omega = 0$, in which case $\boldsymbol{j}_s$ vanishes anyway). The other

term in (11) is of the form of the diffusion theory formula for the current, but apart from ϱ_s replacing ϱ, it contains an additional factor $(2-\omega)$ which is generally larger than unity.

In a purely formal way, one would be inclined to split $\boldsymbol{j}$ in two other parts, namely $(1-\omega)\boldsymbol{j}_d$ which is of the « direct beam » type and $-[(2-\omega)\lambda c/3]\nabla\varrho_s$ which is of the « diffusion » type. This mode of separation is also inspired by the fact that eq. (5) for ϱ_s has the classical appearance of a diffusion equation. Indeed, one easily recognizes the negative second term as the regular absorption term, the third as the source term (with reduced strength), whereas the first one, being of the form

$$D\nabla^2\varrho_s$$

really seems to be derived from a current vector

$$-\frac{(2-\omega)\lambda c}{3}\nabla\varrho_s\,. \tag{12}$$

Now, if (12) is formally regarded as a diffuse contribution to the current, it is rather peculiar that the theory demands this part to be combined with a reduced « direct beam » contribution $(1-\omega)\boldsymbol{j}_d$, precisely as if only a fraction $(1-\omega)$ of the source strength produces a direct beam current. This shows some similarity with the reduction of the source strength in eq. (5) where it looks as if only a fraction ω of the source $S(\boldsymbol{r})$ is responsible for the presence of the full density ϱ_s of scattered particles.

All these particular aspects of the present theory have of course only a formal significance, since they result from an attempt to interpret the various parts of our new formulae in terms of expressions and concepts with which we are familiar through elementary diffusion theory. It does not matter so much to what sort of interpretations the new formulae lead, as long as they form a mathematically consistent theory and yield very accurate results without increasing too much the mathematical difficulties. The first requirement is certainly fulfilled because the only approximation which has been introduced is that ϱ_s is a solution of eq. (5) and this is a consequence of well-established calculations. By the detailed examination of the isotropic point source problem, we have also seen in ref. [1] that the second requirement is quite well fulfilled as far as the density $\varrho(\boldsymbol{r})$ is concerned. In regard to $\boldsymbol{j}(\boldsymbol{r})$, it is equally interesting to make a similar study in a special case in order to know to what degree of accuracy our approximation (7) represents the exact current. This is precisely what we have examined in the next section, in the fundamental case of the isotropic point source. However, before proceeding with these calculations, we first wish to generalize the expression (7) to the

case of slightly non-isotropic scattering, or more precisely, to modify eq. (7) in such a manner that it includes the first degree of non-isotropy in the scattering. To achieve this, we first note that the rigorous relations (1), (2) and (3) remain unchanged when the scattering becomes non-isotropic. This is certainly true for both equations concerning ϱ_a and $\boldsymbol{j}_a$, as these quantities do not depend on the distribution of scattering angles. Furthermore, eq. (1) is known to result from the rigorous Boltzmann equation of the problem in which an integration is carried out over all directions. Finally, one can easily show that in the non-isotropic scattering case, the current $\boldsymbol{j}(\boldsymbol{r})$ can still be represented as a gradient of some scalar function, considering that it results from a superposition of currents from all isotropic sources in which the given isotropic source distribution can be subdivided. Let us now write down the generalized form of eq. (5) which can be deduced from eq. (16) of ref. [1]:

$$\frac{[2-\omega+(1-\omega)^2\,\overline{\cos\gamma}]\lambda v}{3(1-\omega\,\overline{\cos\gamma})}\nabla^2\varrho_s(\boldsymbol{r})-\frac{(1-\omega)v}{\lambda}\,\varrho_s(\boldsymbol{r})+\omega S(\boldsymbol{r})=0\,. \tag{13}$$

This equation can again be combined with (4) and this leads, after the same intermediate steps, to the final formula:

$$\boxed{\boldsymbol{j}\simeq(1-\omega)\boldsymbol{j}_a-\frac{[2-\omega+(1-\omega)^2\,\overline{\cos\gamma}]\lambda v}{3(1-\omega\,\overline{\cos\gamma})}\nabla\varrho_s}\,. \tag{14}$$

This formula becomes again rigorous for $\omega=0$. Only when $\omega=1$ does it give rise to an expression which is completely similar to the corresponding one in elementary diffusion theory (always apart from the fact that ϱ_s replaces ϱ). Indeed in this case we find

$$\boldsymbol{j}=-\frac{\lambda v}{3(1-\overline{\cos\gamma})}\nabla\varrho_s=-\frac{\lambda_{\text{tr}}v}{3}\nabla\varrho_s\,, \tag{15}$$

in which λ_{tr} represents the transport mean free path.

2·2. *Comparison between the exact and the approximate formulae for the particle current* $\boldsymbol{j}$ *in the case of an isotropic point source.* – Since the particle distribution due to an arbitrary given distribution of isotropic sources in an infinite medium can always be regarded as a superposition of point source solutions, there is no loss of generality introduced in our discussion if we choose the isotropic point source problem as the basis for a detailed comparison between

the exact and the approximate particle current $\boldsymbol{j}$. Let us start with the case of isotropic scattering just as we did in our discussion concerning the density. A generalization to non-isotropic scattering can easily be carried out later on.

The isotropic point source located in the origin of our co-ordinate system is described by

$$S_0\,\delta(\boldsymbol{r})\,.$$

In ref. [1], we have found that

$$\varrho(\boldsymbol{r}) = \varrho(r) \simeq \varrho_d(r) + \varrho_s(r)$$

in which

$$\varrho_d(r) = \frac{S_0}{4\pi v}\cdot\frac{\exp\left[-r/\lambda\right]}{r^2}\,,$$

$$\varrho_s(r) = \frac{3\omega S_0}{4\pi(2-\omega)v\lambda r}\exp\left[-\frac{r}{\lambda}\left(3\,\frac{1-\omega}{2-\omega}\right)^{\frac{1}{2}}\right].$$

Let us now calculate $\boldsymbol{j}_d$ and $\nabla\varrho_s$:

(16) $$\boldsymbol{j}_d(r) = \frac{S_0\exp\left[-r/\lambda\right]}{4\pi r^2}\,\mathbf{1}_r\,,$$

(17) $$\nabla\varrho_s = \frac{\mathrm{d}\varrho_s}{\mathrm{d}r}\,\mathbf{1}_r = -\frac{3\omega S_0\mathbf{1}_r}{4\pi(2-\omega)v\lambda r^2}\left(1+K\,\frac{r}{\lambda}\right)\exp\left[-K\,\frac{r}{\lambda}\right],$$

in which $K = \left(3\,\dfrac{1-\omega}{2-\omega}\right)^{\frac{1}{2}}$.

Due to spherical symmetry in the problem, $\boldsymbol{j}$ can be written as

(18) $$\boldsymbol{j} = j_r(r)\mathbf{1}_r\,.$$

Substituting (16) and (17) in (7), our approximation for $j_r(r)$ takes the form:

(19) $$j_r(r) \simeq \frac{S_0}{4\pi r^2}\left\{(1-\omega)\exp\left[-\frac{r}{\lambda}\right] + \omega\left(1+K\,\frac{r}{\lambda}\right)\exp\left[-K\,\frac{r}{\lambda}\right]\right\}.$$

This must be compared with the corresponding rigorous expression which can be deduced from formulae which we published in an earlier paper [3].

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[3] C. C. GROSJEAN: *Nuovo Cimento*, **11**, 11 (1954).

Applying (86) and (73) of this reference to the case of an isotropic point source ($S_1 = 0$), isotropic scattering ($\cos\gamma = 0$) and exponentially decreasing scattering probabilities ($f(l)\,\mathrm{d}l = \omega \exp[-l/\lambda]\,\mathrm{d}l/\lambda$), we get in the present notations:

$$(20)\qquad j_r(r) = \frac{S_0}{4\pi r^2}\left(\exp\left[-\frac{r}{\lambda}\right] + \right.$$
$$\left. + \frac{2\omega}{\pi}\int_0^\infty \frac{\operatorname{arctg} u\,(u - \operatorname{arctg} u)}{(u - \omega \operatorname{arctg} u)}\left(\sin\frac{r}{\lambda}u - \frac{r}{\lambda}u\cos\frac{r}{\lambda}u\right)\frac{\mathrm{d}u}{u^2}\right).$$

The integral appearing in this expression is convergent for all possible values of r and ω, namely $r \geqslant 0$ and $0 \leqslant \omega \leqslant 1$, but it can generally not be worked out in a finite form.

For $\omega = 0$, our approximation is rigorous since both formulae (19) and (20) simply become

$$(21)\qquad j_r(r) = \frac{S_0 \exp[-r/\lambda]}{4\pi r^2}.$$

For $\omega = 1$, our approximation is again exact. Indeed, it yields

$$(22)\qquad j_r(r) = \frac{S_0}{4\pi r^2},$$

whereas the rigorous formula leads to

$$(23)\qquad j_r(r) = \frac{S_0}{4\pi r^2}\left(\exp\left[-\frac{r}{\lambda}\right] + \frac{2}{\pi}\int_0^\infty \operatorname{arctg} u\left(\sin\frac{r}{\lambda}u - \frac{r}{\lambda}u\cos\frac{r}{\lambda}u\right)\frac{\mathrm{d}u}{u^2}\right),$$

in which the integral can be worked out exactly:

$$(24)\qquad \int_0^\infty \operatorname{arctg} u\left(\sin\frac{r}{\lambda}u - \frac{r}{\lambda}u\cos\frac{r}{\lambda}u\right)\frac{\mathrm{d}u}{u^2} = -\int_0^\infty \operatorname{arctg} u\,\mathrm{d}\,\frac{\sin(ru/\lambda)}{u} =$$
$$= \int_0^\infty \frac{\sin(ru/\lambda)}{u}\cdot\frac{\mathrm{d}u}{1+u^2} = \int_0^\infty \frac{\sin(ru/\lambda)}{u}\,\mathrm{d}u - \int_0^\infty \frac{u\sin(ru/\lambda)}{1+u^2}\,\mathrm{d}u.$$

When $r = 0$, both integrals vanish and the expression between the brackets

in (23) is equal to unity. When $r > 0$, we get

$$\int\limits_0^\infty \frac{\sin\,(r/u\lambda)}{u} = \mathrm{d}u\,\frac{\pi}{2}\,, \qquad \int\limits_0^\infty \frac{u\,\sin\,(ru/\lambda)}{1-u^2}\,\mathrm{d}u - \frac{\pi}{2}\exp\left[-\frac{r}{\lambda}\right].$$

and (23) becomes

$$j_r(r) = \frac{S_0}{4\pi r^2}\left\{\exp\left[-\frac{r}{\lambda}\right] + \frac{2}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{2}\exp\left[-\frac{r}{\lambda}\right]\right)\right\} = \frac{S_0}{4\pi r^2}\,. \tag{25}$$

This simple result for the rigorous $j_r(r)$ is not surprising in the case of $\omega = 1$, because in the absence of capture, the net current through any sphere around the point source should be equal to S_0. Let us also mention that (25) satisfies (1) which can now be written as

$$\nabla\cdot(j_r(r)\mathbf{1}_r) = S_0\,\delta(\boldsymbol{r})\,. \tag{26}$$

This explains how it is possible that for $\omega - 1$ the exact formula for the density $\varrho(r)$ still contains an integral which cannot be worked out in a finite form, whereas the sole current component of $\boldsymbol{j}$ is so simple.

For all intermediate values of ω, (19) is of course an approximation. It would again be interesting to compare (19) and (20) numerically for some chosen values of ω, precisely as we did in the case of the density. However due to a simple relation existing between current and density, it appears to be possible to estimate the degree of accuracy of (19) without having to rely upon elaborate computations. Indeed, we know that equation (1) which can now be written as

$$\nabla\cdot\boldsymbol{j} = S_0\,\delta(\boldsymbol{r}) - (1-\omega)v\varrho(r)/\lambda\,, \tag{27}$$

is satisfied by the rigorous as well as by the approximate expressions for $\boldsymbol{j}$ and ϱ. Integrating both sides over a sphere of radius r and applying Green's theorem to the left hand side, we find exactly:

$$4\pi r^2 j_r(r) = S_0 - \frac{4\pi(1-\omega)v}{\lambda}\int\limits_0^r r'^2\varrho(r')\,\mathrm{d}r' = \frac{4\pi(1-\omega)v}{\lambda}\int\limits_r^\infty r'^2\varrho(r')\,\mathrm{d}r'\,, \tag{28}$$

in which the last equality is a consequence of the first eq. (5) of ref. [1]. In this paper we have always examined and tabulated the value sof the

quantity

$$F\left(\frac{r}{\lambda}\right) \equiv \frac{4\pi r^2 v \varrho(r)}{S_0}\,. \tag{29}$$

Introducing this into (28), this formula becomes

$$\frac{4\pi r^2 j_r(r)}{S_0} = 1 - (1-\omega)\int\limits_0^{r/\lambda} F(x)\,\mathrm{d}x = (1-\omega)\int\limits_{r/\lambda}^{\infty} F(x)\,\mathrm{d}x\,. \tag{30}$$

Taking into account that our approximation for (29) does in general not deviate more than 3% from the exact expression and that average deviations taken over intervals of r, in which the deviation does not change sign, are of the order of 2% or less, we may conclude that (19) must be an even more satisfactory approximation for j_r than the one for ϱ, and that it certainly surpasses in accuracy every other approximation of the same degree of simplicity. Our expression (19) is extremely accurate in the vicinity of $\omega = 0$, $\omega = 2 - 12/\pi^2 = 0.78415$ and $\omega = 1$.

In order to compare certain moments of (19) and (20) with each other, the proper Fourier-Bessel transform should be applied to both equations. Considering that (20) can be written as

$$r^{\frac{3}{2}} j_r(r) = \frac{S_0}{(2\pi)^{\frac{3}{2}}}\left\{\left(\frac{\pi}{2r}\right)^{\frac{1}{2}} \exp\left[-\frac{r}{\lambda}\right] + \frac{\omega r}{\lambda}\int\limits_0^{\infty} \frac{\operatorname{arctg}\lambda u\,(\lambda u - \operatorname{arctg}\lambda u)}{(\lambda u - \omega \operatorname{arctg}\lambda u)} J_{\frac{3}{2}}(ru)\frac{\mathrm{d}u}{u_{\frac{1}{2}}}\right\},$$

we obtain by inversion

$$\int\limits_0^{\infty} r^{\frac{3}{2}} j_r(r) J_{\frac{3}{2}}(ru)\,\mathrm{d}r = \frac{S_0}{(2\pi)^{\frac{3}{2}} u^{\frac{1}{2}}}\cdot\frac{(\lambda u - \operatorname{arctg}\lambda u)}{(\lambda u - \omega \operatorname{arctg}\lambda u)}\,,$$

and consequently:

$$\int\limits_0^{\infty} j_r(r)(\sin ru - ru \cos ru)\,\mathrm{d}r = \frac{S_0 u}{4\pi}\cdot\frac{(\lambda u - \operatorname{arctg}\lambda u)}{(\lambda u - \omega \operatorname{arctg}\lambda u)}\,. \tag{31}$$

Applying directly the same transformation to (19), we find

$$\int\limits_0^{\infty} j_r(r)(\sin ru - ru \cos ru)\,\mathrm{d}r \simeq \frac{S_0 u}{4\pi}\left[(1-\omega)\left(1 - \frac{\operatorname{arctg}\lambda u}{\lambda u}\right) + \right.$$
$$\left. + \frac{\omega \lambda^2 u^2}{3((1-\omega)/(2-\omega)) + \lambda^2 u^2}\right]. \tag{32}$$

By series expansion of both right hand sides in powers of λu, one gets:

$$\int_0^\infty j_r(r)(\sin ru - ru\cos ru)\,\mathrm{d}r = \tag{33}$$

$$(a)\qquad = \frac{S_0\lambda^2u^3}{12\pi(1-\omega)}\left[1-\left(\frac{3}{5}+\frac{\omega}{3(1-\omega)}\right)\lambda^2u^2+\dots\right] \qquad \text{(rigorously)},$$

$$(b)\qquad \simeq \frac{S_0\lambda^2u^3}{12\pi(1-\omega)}\left[1-\left(\frac{3}{5}+\frac{\omega}{3(1-\omega)}\cdot\frac{(2-\omega)(1+4\omega)}{5}\right)\lambda^2u^2+\dots\right]$$

(approximately).

This proves that the third order spatial moment of our approximation for $j_r(r)$ is exact:

$$\int_0^\infty r^3 j_r(r)\,\mathrm{d}r = \frac{S_0\lambda^2}{4\pi(1-\omega)}\,. \tag{34}$$

Using (28), this result can also be shown to be a consequence of the second formula (5) in ref. ([1]). All odd higher order moments of (19) are only approximations, but highly accurate ones. This can e.g. be seen for the fifth order moment, comparing (33*a*) and (33*b*) with each other.

The right hand side of (32) appears to be a good approximation for small values of λu, but it is also interesting to note that it is at the same time an equally good approximation for very large values of λu. Indeed, expanding both right hand sides of (31) and (32) in power series of $1/\lambda u$, we find:

$$\int_0^\infty j_r(r)(\sin ru - ru\cos ru)\,\mathrm{d}r = \tag{35}$$

$$(a)\qquad = \frac{S_0u}{4\pi}\left[1-\frac{(1-\omega)\pi}{2}\cdot\frac{1}{\lambda u}+(1-\omega)\left(1-\frac{\pi^2\omega}{4}\right)\frac{1}{\lambda^2u^2}+\dots\right] \qquad \text{(rigorously)},$$

$$(b)\qquad \simeq \frac{S_0u}{4\pi}\left[1-\frac{(1-\omega)\pi}{2}\cdot\frac{1}{\lambda u}+(1-\omega)\left(1-\frac{3\omega}{(2-\omega)}\right)\frac{1}{\lambda^2u^2}+\dots\right]$$

(approximately) $(\lambda u \gg 1)$.

Finally, it can be stated that (19) has the correct behaviour in the region around the source. Indeed, (20) and (19) respectively lead to the following

expansions for small values of r:

$$(36)\quad \begin{cases} (a) \quad 4\pi r^2 j_r(r) = S_0\left[1-(1-\omega)\dfrac{r}{\lambda}+\dfrac{(1-\omega)}{2}\left(1-\dfrac{\pi^2\omega}{4}\right)\dfrac{r^2}{\lambda^2}+\ldots\right], \\ (b) \quad 4\pi r^2 j_r(r) \cong S_0\left[1-(1-\omega)\dfrac{r}{\lambda}+\dfrac{(1-\omega)}{2}\left(1-\dfrac{3\omega}{(2-\omega)}\right)\dfrac{r^2}{\lambda^2}+\ldots\right]. \end{cases}$$

The marked similarity between the expansions (35) and (36) is a consequence of the existence of certain Abelian and Tauberian theorems relating the behaviour of a function and its transform.

From this detailed analysis, it can be concluded that our approximation (19) is highly accurate for all values of the parameters and that it has remarkable properties.

Let us now apply (19) to the general case of an arbitrary given distribution of isotropic sources described by $S(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'$. This leads directly to the following integral representation of $\boldsymbol{j}(\boldsymbol{r})$:

$$(37)\quad \boldsymbol{j}(\boldsymbol{r}) \cong \frac{(1-\omega)}{4\pi}\iiint_{-\infty}^{\infty} S(\boldsymbol{r}')\,\frac{(\boldsymbol{r}-\boldsymbol{r}')\exp[-|\boldsymbol{r}-\boldsymbol{r}'|/\lambda]}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}\boldsymbol{r}' + \\ -\frac{\omega}{4\pi}\iiint_{-\infty}^{\infty} S(\boldsymbol{r}')\,\frac{(\boldsymbol{r}-\boldsymbol{r}')[1+(K|\boldsymbol{r}-\boldsymbol{r}'|/\lambda)]\exp[-K|\boldsymbol{r}-\boldsymbol{r}'|/\lambda]}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}\boldsymbol{r}'.$$

Taking (8) into account and considering the result obtained for $\varrho_s(\boldsymbol{r})$ in ([1]), namely

$$(38)\quad \varrho_s(\boldsymbol{r}) = \frac{1}{4\pi v\lambda}\cdot\frac{3\omega}{(2-\omega)}\iiint_{-\infty}^{\infty} S(\boldsymbol{r}')\,\frac{\exp[-K|\boldsymbol{r}-\boldsymbol{r}'|/\lambda]}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\boldsymbol{r}', \qquad \left[K=\left(3\,\frac{1-\omega}{2-\omega}\right)^{\frac{1}{2}}\right],$$

one can easily show after having calculated the gradient of (38), that (37) can be written as

$$\boldsymbol{j}(\boldsymbol{r}) \cong (1-\omega)\boldsymbol{j}_d(\boldsymbol{r}) - \frac{(2-\omega)\lambda v}{3}\nabla\varrho_s(\boldsymbol{r}).$$

This does not only constitute another proof of the correctness of eq. (7) but it is also a proof of its validity. Indeed by showing that (7) and (37) are identical and considering that (37) derives in fact from a superposition of point source solutions, it is evident that (7) must always be a high

accuracy approximation having many properties in common with the corresponding rigorous particle current.

The preceding analysis can be completely repeated in the case of non-isotropic scattering in which the angular distribution of the secondaries after a collision is described by the following probability function:

$$p(\gamma)\,\frac{\sin\gamma\,\mathrm{d}\gamma}{2} = (1 + 3\,\overline{\cos\gamma}\,\cos\gamma)\,\frac{\sin\gamma\,\mathrm{d}\gamma}{2}\,.$$

Applying our formula (14) to the point source problem, we find exactly the same expression as (19) for $j_r(r)$, but with a corrected factor K, namely

$$j_r(r) \cong \frac{S_0}{4\pi r^2}\left\{(1-\omega)\exp\left[-\frac{r}{\lambda}\right] + \omega\left(1 + K\frac{r}{\lambda}\right)\exp\left[-K\frac{r}{\lambda}\right]\right\}, \tag{39}$$

in which

$$K = \left(\frac{3(1-\omega)(1-\omega\,\overline{\cos\gamma})}{[2-\omega+(1-\omega)^2\,\overline{\cos\gamma}]}\right)^{\frac{1}{2}}. \tag{39'}$$

This result has to be compared with the rigorous expression for $j_r(r)$ which we can again deduce from the formulae (86) and (73) of ref ([3]):

$$j_r(r) = \frac{S_0}{4\pi r^2}\left\{\exp\left[-\frac{r}{\lambda}\right] + \frac{2\omega}{\pi}\cdot\right. \tag{40}$$

$$\left.\cdot\int_0^\infty \frac{\left[\operatorname{arctg} u + \dfrac{3(1-\omega)\,\overline{\cos\gamma}}{u}\left(1-\dfrac{\operatorname{arctg} u}{u}\right)\right](u - \operatorname{arctg} u)}{u - \omega\operatorname{arctg} u - \dfrac{3\omega(1-\omega)\,\overline{\cos\gamma}}{u}\left(1-\dfrac{\operatorname{arctg} u}{u}\right)}\left(\sin\frac{r}{\lambda}u - \frac{r}{\lambda}u\cos\frac{r}{\lambda}u\right)\frac{\mathrm{d}u}{u^2}\right\}.$$

The conclusions remain entirely unchanged:

— (39) is rigorous for $\omega = 0$ and for $\omega = 1$, and the formulae (21) and (25) are unaffected by the non-isotropy of the scattering;

— for $0 < \omega < 1$, our formula is a very accurate approximation and (28) remains valid;

— the third order moment of (39) is exact and takes the form

$$\int_0^\infty r^3 j_r(r)\,\mathrm{d}r = \frac{S_0\lambda^2}{4\pi(1-\omega)(1-\omega\,\overline{\cos\gamma})}, \tag{41}$$

whereas all other moments are accurate approximations;

— in the region around the source, the expansion of $4\pi r^2 j_r(r)$ is again rigorous up to the first power of r/λ.

From all these calculations, it appears quite clearly that in the present paper we have succeeded in completing our new formalism in the sense that we have added to it the possibility of obtaining a high accuracy approximation for the particle current in each considered case. We have shown that the general expressions (7) and (14), being equivalent to superpositions of point source solutions, will always lead to results having important characteristics in common with the exact current, although they appear in a relatively simple form. Let us also remember that together with the other equations of the theory, (7) and (14) form respectively in the isotropic and in the non-isotropic scattering case, an entirely consistent formalism.

* * *

We wish to express our sincere thanks to the Institut Interuniversitaire des Sciences Nucléaires and to Prof. Dr. J. L. VERHAEGHE for his continuous interest in our research work.

RIASSUNTO (*)

Nel quadro di un nuovo formalismo approssimato per la soluzione di problemi di scattering multiplo ad una velocità in mezzi omogenei infiniti, presentato in un precedente lavoro, si derivano e si discutono dettagliatamente le adatte espressioni per la corrente di particelle nei due casi di scattering isotropico e leggermente non isotropico. Si esegue un confronto dettagliato tra le corrispondenti formule rigorose ed approssimate per il caso speciale di una sorgente puntiforme isotropica, giungendo alla conclusione che le nuove approssimazioni ottenute sono altamente accurate e presentano notevoli proprietà.

(*) *Traduzione a cura della Redazione.*

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On the Real Part of the Complex Potential Well of the Nucleus.

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(ricevuto il 20 Giugno 1956)

Summary. — Starting from a well-defined assumption about nuclear forces, the real part of the complex potential which enters into the description of the neutron-nucleus interaction is determined for energies of the incoming neutron ranging from 0 to 400 MeV. The assumption is made that nuclear matter remains unperturbed as the neutron passes through it. The exchange character of nuclear forces is essential for the results obtained.

In a note by Kind and Villi (1) (later indicated as I) some results have been given of a calculation, the purpose of which was to determine the real part of the complex potential of the nucleus starting from well defined assumptions about the interaction between free nucleons.

The purpose of the present note is to give the results of a more complete calculation on the same subject, and to discuss these results.

1. – Experimental data.

The real part V of the complex potential, which enters into the optical model description of the neutron-nucleus interaction, can be determined experi-

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(1) A. Kind and C. Villi: *Nuovo Cimento*, **1**, 749 (1955).

mentally from the analysis of the total neutron cross-sections as functions of the energy E of the incoming neutron. For the limiting case $E = 0$, the value of V has been determined by ADAIR (2) and by FESHBACH, PORTER, and WEISSKOPF (3) to be nearly 42 MeV.

For $50 < E < 400$ MeV the potential V has been analyzed by T. B. TAYLOR (4).

2. – Interaction between Nucleons.

Very little can be said to-day about the real features of nuclear forces and especially about the relation between the forces acting between two free nucleons and those acting in the interior of the nucleus.

In any case it must, however, be admitted that n-body forces may play a role in the nucleus. They might, at least partly, be responsible for the saturation of the binding energy.

About the n-body forces it can, for instance, be said that for different n's they will be affected by the exclusion principle in a different way. One can therefore foresee that for the interaction of one nucleon with the nucleus they will depend in different ways on the velocity of the incident particle.

Our knowledge of this and other questions about the nuclear forces are, however, to-day so poor that the best thing we can do is to start with the simplest possible assumption. On the basis of the results obtained we might then try to draw some conclusions on more detailed questions. To do this we assume that nuclear forces in the nucleus are the same as those acting between two free nucleons, and that they are given by the interaction Hamiltonian

$$H_{\text{int}}(1, 2) = -\frac{\exp[-r/\beta]}{r/\beta} W(1 + \alpha P^M), \tag{1}$$

where r is the distance between the two nucleons, and P^M is the space co-ordinates exchange operator. We choose W and β so as to fit correctly the experimental low energy behaviour of the two nucleons systems:

$$\mathrm{W}(1 + \alpha) = 57.9 \text{ MeV} \tag{2}$$

$$\beta = 1.25 \cdot 10^{-13} \text{ cm} \quad (5). \tag{3}$$

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(2) R. K. ADAIR: *Phys. Rev.*, **94**, 737 (1954).
(3) H. FESHBACH, C. E. PORTER and V. F. WEISSKOPF: *Phys. Rev.*, **96**, 448 (1954).
(4) T. B. TAYLOR: *Phys. Rev.*, **92**, 831 (1953).
(5) W. WILD and K. WILDERMUTH: *Zeits. f. Naturfor.*, **9a**, 799 (1954).

Thereby the ratio α between the Majorana and Wigner parts of the potential is not determined.

We shall see that this ratio plays an essential role in the description of the neutron-nucleus interaction.

3. – The Weak Incoherent Interaction Approximation.

The assumption that the interaction Hamiltonian H_{int} can be written in the form (1) makes it possible to calculate the potential $V(E)$ using the weak incoherent interaction approximation as described in a note of one of the authors (6). We shall stop here at the zero order approximation.

For low energies the validity of such an approach is essentially supported by the effect of the exclusion principle which strongly inhibits transitions in the nucleus (7).

If the energy E of the incoming nucleon is increased, the exclusion principle becomes rapidly less effective and correlation effects become more important. The convergence of the method becomes worse.

However, if the energy E is increased to such a value that the interaction between two nucleons can be correctly treated in the Born approximation, it is again correct to determine $V(E)$ in terms of the weak incoherent interaction approximation, i.e. by assuming that nuclear matter remains nearly unchanged when the nucleon passes through it.

4. – Calculation of $V(E)$.

The main lines of the calculation of $V(E)$ have been given in I. In that paper, however, the assumption has been made of no neutron excess in the nucleus. Furthermore it must be pointed out that, because of a numerical error, the quantity $V(k')$ calculated in I is too big. This error has been corrected in the present calculation. Here, as in I, we assume that the nuclear radius is

$$R = r_0 \cdot A^{\frac{1}{3}} \qquad \text{with} \qquad r_0 = 1.43 \cdot 10^{-13}\ \text{cm}\,, \tag{4}$$

and we take into account a neutron excess by putting

$$\frac{2Z}{A} = 0.82\,, \tag{5}$$

where Z and A are the numbers of protons and nucleons, respectively.

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(6) A. Kind: *Nuovo Cimento*, **2**, 443 (1955).
(7) V. F. Weisskopf: *Helv. Phys. Acta*, **23**, 187 (1950).

Nuclear matter is described by means of a product of plane wave functions φ_j, with wave vector k_j, antisymmetrized with the plane wave function φ' describing the incident neutron, whose wave vector inside the nucleus is k'.

In the approximation which we have chosen, the depth V of the potential well felt by the neutron inside the nucleus is

$$V(k') = \sum_j \int \varphi_j^*(q_1)\varphi'^*(q_2)H_{\text{int}}(q_1, q_2)[\varphi_j(q_1)\varphi'(q_2) - \varphi_j(q_2)\varphi'(q_1)]\, \mathrm{d}q_1\, \mathrm{d}q_2 \,, \tag{6}$$

where the summation is extended over all states φ_j. The q_i represent the space, spin, and isotopic spin co-ordinates of the nucleons.

By dividing the integral on the right-hand side of (6) in its direct and exchange parts we get

$$V(k') = I_{\text{d}} - I_{\text{ex}} \tag{7}$$

with

$$I_{\text{d}} = - W\left[\frac{2Z}{A}(\alpha - 4) + \frac{2(A-2Z)}{A}(\alpha - 2)\right]\frac{2}{3}\,\frac{k^3}{\pi^0}\,, \tag{8}$$

and

$$I_{\text{ex}} = W\left[\frac{2Z}{A}\,\frac{16\alpha - 4}{4} + \frac{2(A-2Z)}{A}(2\alpha - 1)\right]\frac{1}{\pi}\,G(k_0, k')\,, \tag{9}$$

where k_0 is the radius of the Fermi sphere in units β^{-1}, and $G(k_0, k')$ is the function given by (6) in I.

Finally from (7) and

$$E[(k'/k)^2 - 1] = V(k') \tag{10}$$

(see formulae (2) and (7) of I), we get V as a function of the kinetic energy of the neutron outside the nucleus.

5. – Results and Discussion.

In Fig. 1 the empirical data are indicated by the cross (+) and the dotted curve. The theoretical results are reproduced by the three full curves obtained by putting $\alpha = 1$, 2.3, and 4. The value $\alpha = 2.3$ has been chosen to fit the empirical data at high energy ($E = 400$ MeV). At this value of E and with the H_{int} given by (1), one has $W(1+\alpha)\beta/\hbar v = 0.4$ (v is the velocity of the incident nucleon). This means that in this case it begins to be reasonabe to treat the two nucleons interaction in the Born approximation and, therefore, also our expression (6) for $V(k')$ is significant. The value $\alpha = 2.3$ is, however, not in accord with the high energy neutron-proton scattering, which requires

a value of α near to 1. On the other hand, the curve for $\alpha = 1$ lies much too high with respect to the empirical data.

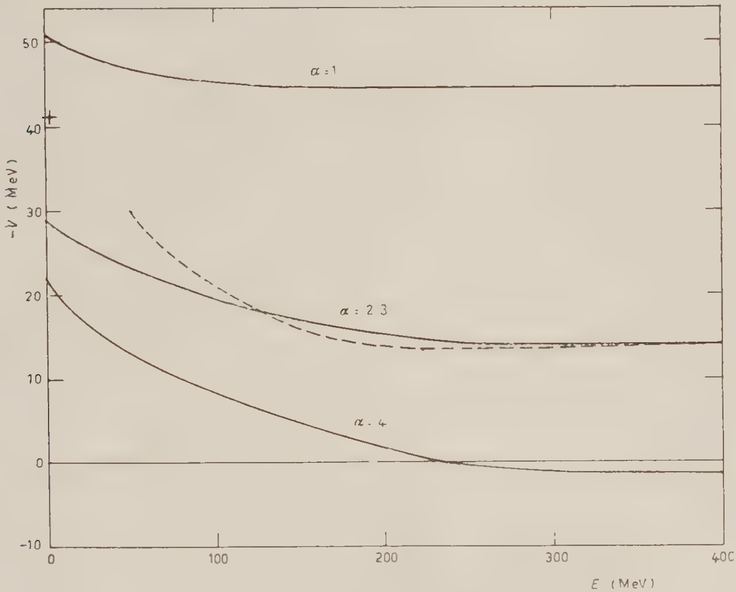

Fig. 1.

So, although there is a qualitative agreement between theoretical and empirical data (and we emphasize that this essentially depends on the exchange term (9)), we must conclude that there are some essential discrepancies between the theoretical and the empirical results which cannot depend only on the approximation (6) which we used, but must essentially depend on the Hamiltonian (1) which is not the correct one.

We know that, to fit the two-body experimental data, it is necessary to take into account a tensor force and to introduce also a repulsive core in the interaction between two nucleons. Both of these corrections are such as to modify our theoretical results toward a better accord with the empirical data.

In fact, with the introduction of a tensor force, it is possible to fit correctly the two-body experimental data by reducing the strength W of the central force [8]. In our expression (6), which would continue to be valid for the

(8) For example, with the potential mentioned in the book of BLATT and WEISSKOPF: *Theoretical Nuclear Physics*, at p. 103, the strength of the central force is reduced to about $\frac{1}{3}$ of the value we have used here.

limiting case of high energies, the tensor force would give a zero contribution, and the curve for $\alpha = 1$ would get deeper.

The presence of a repulsive core would diminish the interaction between nucleons essentially in case of higher energies and would therefore deform the curve for $\alpha = 1$ so as to fit the empirical data better [9].

The correlation effects are not contained in the approximation (6). These would modify the curve $V(E)$ essentially in the middle energy region. However, it has to be emphasized that, as shown in I, $V(E)$ is not equal to the total work done by nuclear forces during the entrance of the neutron into the nucleus, but only equal to the part of this work which effectively serves to increase the mean kinetic energy of the incoming neutron. The rest of this work is elastically absorbed by the virtual excitation of nuclear matter. Therefore, although correlations imply an augmentation of the interaction energy between neutron and nucleus, it is, nevertheless, not possible to foresee their effect on $V(E)$ without the performance of an exact analysis of the problem.

* * *

The authors wish to thank Dr. B. MOTTELSON for useful discussions on the subject.

(9) See also R. JASTROW: *Phys. Rev.*, **82**, 261 (1951).

RIASSUNTO

Partendo da una data interazione nucleone-nucleone si determina la parte reale del potenziale complesso che descrive l'interazione neutrone-nucleo, per energie comprese fra 0 e 400 MeV. Il calcolo è effettuato nell'ipotesi che la materia nucleare si possa considerare imperturbata durante l'interazione col neutrone incidente. Il carattere di scambio delle forze nucleari è essenziale per i risultati che si ottengono.

On the Validity of the Williams-Weizsäcker Method and the Problem of the Nuclear Interactions of Relativistic μ-mesons.

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(ricevuto il 22 Giugno 1956)

Summary. — After discussing arguments which have been put forward against the application of the semi-classical Williams-Weizsäcker method to the problem of the nuclear interactions of relativistic μ-mesons, we develop a new treatment consistent with quantum field theory. Our results agree with Williams' and Weizsäcker's, in the sense that the μ-meson and photon cross-sections can be compared for corresponding processes by means of an « equivalent photon spectrum », almost the same in both theories. It is shown that in those non-elastic photonuclear processes the angular deviation of the μ-meson is not uniquely related to primary energy and energy-transfer. It is in most cases of practical interest much smaller than in the elastic Coulomb scattering.

1. – Introduction.

Since the discovery of the nuclear interactions of high energy μ-mesons by George *et al.* [1], the interpretation of this phenomenon in terms of photonuclear interactions was generally accepted in view of the fact that the experimental results were in reasonable agreement with the predictions of the well-known semi-classical Williams-Weizsäcker method [2,3] (*). We recall that the

(1) E. P. George and J. Evans: *Proc. Phys. Soc.* (*London*), A **63**, 1248 (1950); E. P. George: *Progress in Cosmic Ray Physics* (Amsterdam, 1952), chap. VII.

(2) E. J. Williams: *Proc. Roy. Soc.*, A **139**, 163 (1933).

(3) C. F. v. Weizsäcker: *Zeits. f. Phys.*, **88**, 612 (1934).

(*) Henceforth referred to as WW method.

idea of this latter theory is to decompose the electromagnetic field produced by the μ-meson at the target into a spectrum of equivalent photons, so that the cross-section of the interaction of the μ-meson can be obtained by integration of the experimentally known photon cross-section over this spectrum. In order that this method should be applicable, the μ-meson must be relativistic (i.e. $E \gg mc^2$) and the energy-loss or energy of the virtual light quantum q should be small compared to the total energy of the μ-meson ($q \ll E$). Those conditions are fulfilled for μ-mesons producing stars in nuclear emulsions or other detecting devices underground. With the above assumptions, the virtual spectrum of the μ-meson can be written (⁴), as follows:

$$N(q)\,\mathrm{d}q = (2\alpha/\pi)\,\mathrm{d}q/q\cdot\ln\,(E/aq - 0.38)\,, \tag{1}$$

where α is the fine structure constant and « a » is a constant of the order unity.

Recently however, FOWLER (⁵) questioned the validity of the application of the WW method to the present problem. His point is that in the case of stars produced by μ-mesons, where $q \gg \hbar/R$ (q is the transferred momentum, and R the linear dimension of the target nucleus), the equivalent photon spectrum decreases faster than predicted by WW (it goes as $q^{-\frac{3}{2}}\,\mathrm{d}q$ instead of $q^{-1}\,\mathrm{d}q$), so that the total cross-section comes out about 30 times smaller than the observed one. FOWLER suggests therefore the existence of a new kind of interaction between μ-meson and nucleon.

Now, although it is clear that the validity of the semi-classical picture of WW should not be taken for granted without further investigation, it seems that FOWLER's arguments are not pertinent for the following reasons:

1) The use of Møller potentials as well as the relation $q \sim p\theta$ (p = momentum of the μ-meson, θ = angular deviation) apply only to the case of elastic scattering, where the μ-meson knocks on elastically one of the protons of the target nucleus and the recoil momentum of the proton alone produces the star (in star production, the energy transfer is much larger than the binding energy of the nucleons inside the target nucleus, so that they can be considered as free). It is well known however that pure Coulomb interactions can explain only a small fraction of the experimental cross-sections (see e.g. JAHN (⁶)). Fowler's treatment cannot be applied to more complicated interactions, which seem to be responsible for the bulk of the observed cross-

(⁴) W. HEITLER: *The Quantum Theory of Radiation*, 3-rd edition (Oxford, 1954), p. 414 ff.

(⁵) G. N. FOWLER: *Proc. Phys. Soc.* (*London*), A **68**, 482 (1955) and *Nucl. Phys.*, **1**, 119 (1956).

(⁶) H. JAHN: in *Kosmische Strahlung*, 2-nd edition (Berlin, 1953), chap. III, 5.

sections (such as meson production where the meson either comes out as a free particle or is reabsorbed in the parent nucleus).

2) Moreover, if only elastic scattering is considered, one should observe that the very idea of the WW method is *a priori* not applicable, since no corresponding photon cross-section exists for this process, as real photons are not absorbed at first order by free protons. The observed photonuclear cross-section of about 10^{-28} cm²/nucleon belongs to entirely different processes, and therefore should not be introduced into the *elastic* scattering cross-section of μ-mesons (*).

In the following investigation we shall treat the problem of the interaction of relativistic μ-mesons from the point of view of field theory. This treatment is in fact more general and can be applied to any electromagnetic interaction. It will be seen that Williams' and Weizsäcker's results are valid from the new point of view. Furthermore, we shall exhibit some new results concerning the angular deviation of the μ-meson, which seem to have been overlooked hitherto by the users of the WW method.

2. – The Equivalent Photon Spectrum of the Relativistic μ-Meson.

The purpose of this calculation is to express the nuclear interaction of the μ-meson (+) as follows:

$$\sigma_\mu(E) = \int N(q, E)\, \sigma_{\text{ph}}(q)\, \mathrm{d}q \,. \tag{2}$$

(*) Fowler seems to have attached much importance to the distinction between small and large q compared to $\hbar/R$. However, from a quite general argument one should not expect anything special to happen to the equivalent photon-spectrum when the photon wave length becomes smaller than the nuclear dimensions. It may be that the cross-section then decreases faster, but exactly the same would happen also to the cross-section of free photons, so that the effect is expected to cancel when the two are compared as in the WW method. Thus the equivalent photon spectrum is expected to be smooth and as one uses the experimentally given (and theoretically extrapolated) photon cross-section, the case of small wave lengths needs no separate consideration (at least as long as the momentum transfer remains small compared to the momentum of the primary). The above distinction is indeed not fundamental. In Fowler's work it corresponds to the fact that the momentum-space available to the outgoing μ-meson in Coulomb scattering is reduced by energy-momentum conservation when the energy-transfer becomes large enough so that the nucleons can be considered as free. But apparently it has been overlooked that the corresponding photon cross-section then goes to zero by the same argument.

(+) The treatment is of course valid also for other processes (e.g. electron-positon pair-production) but we shall keep in the following the expression « nuclear interaction » for simplicity of language.

Here $\sigma_{\mu}(E)$ is the cross-section of a μ-meson of energy E for a specified process, $\sigma_{\text{ph}}(q)$ that of a photon of energy q *for the same process* and $N(E, q)$ a factor depending on E and q which can be considered as the equivalent photon spectrum of the meson, provided it is independent of the particular process considered. We consider only photoelectric interactions of the μ-meson with the nucleus to lowest order (one photon exchanged). Thus we wish to compare the cross-sections corresponding to the two Feynman graphs I and II (Fig. 1), where $\mathfrak{M}$ is the representative of the rest of the graph, which may be as complicated as one wants, as long as energy and momentum are conserved at each vertex and *the same* $\mathfrak{M}$ is used in both graphs.

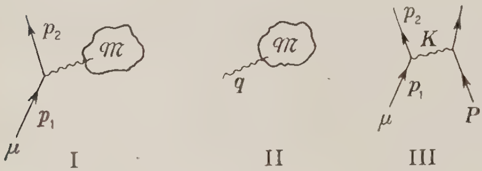

Fig. 1.

When one goes over to overall cross-sections (i.e. for all nuclear events above a certain energy) one should remember the fact mentioned in the introduction. The total cross-section of the μ-meson includes elastic Coulomb scattering on the protons of the nucleus (considered as free at the energies of stars, $\gtrsim 150$ MeV), graph III (Fig. 1), for which process there exists no photon counterpart and for which no formula like (2) can be derived. Therefore we shall have

$$\sigma_{\mu}^{\text{tot}}(E) = \int N(q, E)\, \sigma_{\text{ph}}^{\text{tot}}(q)\, \mathrm{d}q + \sigma_{\mu}^{\text{elastic}} . \tag{3}$$

The matrix element corresponding to the Feynman graph I which represents any particular interaction of the μ-meson, can be written as follows:

$$\tau_{\mu}(\boldsymbol{p}) = \frac{-ie}{\hbar c} \cdot \frac{-i\hbar c}{(2\pi)^4} \int \frac{1}{(2\pi)^3} \frac{m}{\sqrt{E_1 E_2}} \frac{1}{\boldsymbol{k}^2} \cdot \tag{4}$$
$$\cdot [\overline{w}(2)\gamma_{\nu} w(1)] \mathfrak{M}^{\nu} (2\pi)^4 \delta^{(4)}(\boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{k}) \cdot (2\pi)^4 \delta^{(4)}(\textstyle\sum \boldsymbol{p}_i + \boldsymbol{k} - \sum \boldsymbol{p}_f)\, \mathrm{d}^4 k .$$

Here $w(1)$ is written for the Dirac spinor $w^{s_1}(\boldsymbol{p}_1)$ representing the μ-meson with spin direction s_1 and energy-momentum four-vector $\boldsymbol{p}_1$ ($\vec{p}_1$, $E_1 = \sqrt{p_1^2 + m^2}$) in the initial state, and likewise $w(2)$ represents the μ-meson in the final state. The meson current $\overline{w}(2)\gamma_{\nu} w(1)$ is coupled with $\mathfrak{M}^{\nu}$ (the summation must be carried out over the dummy indices). $\mathfrak{M}^{\nu}$ is the representative of the contribution of all particle lines and vertices of the remaining part of the particular graph. The index ν of $\mathfrak{M}^{\nu}$ belongs to the four-vector appearing at the vertex where the photon ends (fermion current or boson momentum). $1/\boldsymbol{k}^2$ is the photon propagator. The delta functions insure energy and momentum-con-

servation ($\boldsymbol{k}$ is the energy-momentum four-vector of the photon, $\sum \boldsymbol{p}_i$ and $\sum \boldsymbol{p}_f$ are respectively the total initial and final energy-momentum four-vectors of all incoming or outgoing particles except the μ-meson).

Integrating equation (4) with the help of the delta function, one gets

$$(5)\qquad \tau_\mu(\boldsymbol{p}) = \frac{-e}{(2\pi)^3}\frac{m}{\sqrt{E_1E_2}}\frac{1}{(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}[\overline{w}(2)\gamma_\nu w(1)]\mathcal{M}^\nu\cdot$$
$$\cdot(2\pi)^4\delta^{(4)}(\sum \boldsymbol{p}_i+\boldsymbol{p}_1-\sum \boldsymbol{p}_f-\boldsymbol{p}_2)\,.$$

From this one gets the cross-section in the usual way by squaring the matrix element, dividing by the flux $c/(2\pi)^3$ of the incoming μ-meson (the relativistic velocity has been taken equal to c) and by a factor $(2\pi)^4\delta^{(4)}(0)$ (which is interpreted as VT, the product of the large volume and time interval in which the interaction takes place) and integrating over the final states of the μ-meson. The division by the density of the target particles as well as integration over the final states of all particles except the μ-meson, will be expressed formally by replacing $\mathcal{M}^\nu$ by $\mathcal{N}^\nu$, so that finally

$$(6)\qquad \sigma_\mu(\boldsymbol{p}_1) = \frac{e^2}{c(2\pi)^3}\int\frac{m^2}{E(p_1)E(p_2)}\frac{p_2^2\,\mathrm{d}p_2\,\mathrm{d}\Omega}{(\boldsymbol{p}_1-\boldsymbol{p}_2)^4}\cdot$$
$$\cdot|[\overline{w}(2)\gamma_\nu w(1)]\mathcal{N}^\nu|^2(2\pi)^4\delta^{(4)}(\sum \boldsymbol{p}_i+\boldsymbol{p}_1-\sum \boldsymbol{p}_f-\boldsymbol{p}_2)\,.$$

For the Feynman graph II representing the corresponding photo-nuclear interaction,

$$(7)\qquad \tau_{\text{ph}}(\boldsymbol{q}) = \frac{(\hbar c)^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2q_0}}\,\varepsilon_\nu\mathcal{M}^\nu(2\pi)^4\delta^{(4)}(\sum \boldsymbol{p}_i+\boldsymbol{q}-\sum \boldsymbol{p}_f)\,.$$

Here $\boldsymbol{q}$ is the energy-momentum four vector ($\vec{q}$, $q_0=|\vec{q}|$) of the free photon, ε^ν is the polarization four-vector (where we have dropped a superscript referring to a particular polarization since no confusion can occur). $\mathcal{M}^\nu$ is exactly the same as above in the case of graph I.

From (7) we get the cross-section as above, except that there is now no integration over final particle states other than those already included in $\mathcal{N}^\nu$.

$$(8)\qquad \sigma_{\text{ph}}(\boldsymbol{q}) = (\hbar/2q_0)\,|\varepsilon_\nu\mathcal{N}^\nu|^2(2\pi)^4\delta^{(4)}(\sum \boldsymbol{p}_i+\boldsymbol{q}-\sum \boldsymbol{p}_f)\,.$$

We shall now transform the μ-meson cross-section (6) into an expression of the form (2). Let us consider first the factor $p_2^2\mathrm{d}p_2\mathrm{d}\Omega/(\boldsymbol{p}_1-\boldsymbol{p}_2)^4$. We write $\mathrm{d}\Omega=2\pi\sin\theta\,\mathrm{d}\theta$, where θ is the angle of deviation of the μ-meson. On the other hand

$$(9)\qquad (\boldsymbol{p}_1-\boldsymbol{p}_2)^4=[(\vec{p}_1-\vec{p}_2)^2-(E_1-E_2)^2]^2=$$
$$=p_1^2p_2^2[4\sin^2(\theta/2)+m^2(p_1-p_2)^2/p_1^2p_2^2]^2\,,$$

where $p_1 = |\vec{p}_1|$ and $p_2 = |\vec{p}_2|$ and neglecting terms of higher order in m^2/p_1p_2.

It appears a priori from (9) that the main contribution of $1/(\boldsymbol{p}_1 - \boldsymbol{p}_2)^4$ to the integral (6) will come from small values of the quantity $(p_1 - p_2)/p_1$ and very small values of θ (of the order of $m(p_1 - p_2)/p_1p_2$).

We introduce the following notations: $\mu = m/E_1$; $\varkappa = (p_1 - p_2)/p_1$; $\varphi = \theta/\varkappa$. These three quantities will be treated as small: $\mu \ll 1$ because the μ-meson is considered as relativistic, $\varkappa \ll 1$ for the reason we have just mentioned following equation (9) and finally $\varphi \sim \mu$ as we have seen above that $\theta \sim \mu\varkappa$.

Finally we get, neglecting terms of higher order in μ, $\varkappa$ and φ

(10)
$$\frac{p_2^2\,\mathrm{d}p_2\,\mathrm{d}\Omega}{(\boldsymbol{p}_1 - \boldsymbol{p}_2)^4} = -2\pi\cdot\frac{\mathrm{d}\varkappa}{p_1\varkappa^2}\,\frac{\varphi\,\mathrm{d}\varphi}{(\varphi^2+\mu^2)^2}\,.$$

Let us now calculate the components of the four-vector $j_\nu = (m/\sqrt{E_1E_2})\cdot\cdot\overline{w}(2)\gamma_\nu w(1)$. We shall adopt the orthogonal coordinate system $Oxyz$ where the x axis is defined by $\vec{p}_1 - \vec{p}_2$ and the y axis is perpendicular to both $\vec{p}_1$ and $\vec{p}_2$ (Fig. 2). Note that the angle between $\vec{p}_1$ and the x axis is equal to φ neglecting higher order corrections.

The calculations are trivial. One gets, neglecting terms of higher order than the second in μ, $\varkappa$ and φ:

Case without spin flip $(s_1 = s_2)$

$$j_x = 1 - \varphi^2/2 - \mu^2/2$$
$$j_y = -\,i\varkappa\mu/2$$
$$j_z = \varphi + \varkappa\varphi/2$$
$$j_0 = 1.$$

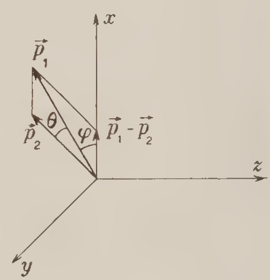

Fig. 2.

Case with spin-flip $(s_1 \neq s_2)$

$$j_x = \varkappa\varphi/2$$
$$j_y = -\,i\varkappa\varphi/2$$
$$j_z = \varkappa\mu/2$$
$$j_0 = \varkappa\varphi/2\,.$$

If we neglect now also the second order terms, one sees that we may restrict ourselves to the case without spin-flip. Thus we have

(11)
$$\boldsymbol{j} = (1,\ 0,\ \varphi,\ 1).$$

We shall put $\boldsymbol{j} = \varphi\boldsymbol{\epsilon}'$ with $\boldsymbol{\epsilon}' = (\varepsilon_0',\ 0,\ 1,\ \varepsilon_0')$ and $\varepsilon_0' = 1/\varphi$. Inserting this

and (10) into (6), we get the following expression for the cross-section:

$$\sigma_\mu(\boldsymbol{p}_1) = \frac{e^2}{4\pi^2 c}\int \frac{\mathrm{d}\varkappa}{\varkappa^2 p_1}\,\frac{\varphi^3\,\mathrm{d}\varphi}{(\varphi^2+\mu^2)^2}\,|\varepsilon'_\nu \mathcal{T}^\nu|^2(2\pi)^4\,\delta^{(4)}\left(\sum \boldsymbol{p}_i + \boldsymbol{p}_1 - \sum \boldsymbol{p}_f - \boldsymbol{p}_2\right). \tag{12}$$

Now we have to show a) that the emitted photon can be assimilated to a real photon with an energy momentum four-vector $\boldsymbol{q} = \boldsymbol{p}_1 - \boldsymbol{p}_2$ and b) that $\boldsymbol{\epsilon}'$ can be assimilated to the polarization four-vector $\boldsymbol{\epsilon}$ of a real photon.

a) It is sufficient to show that the condition $q_0 = \vec{q}$ is satisfied (up to an approximation of higher order) by the virtual photon. Now one has

$$\frac{|\vec{q}\,|^2 - q_0^2}{p_1 p_2} = \frac{(\vec{p}_1 - \vec{p}_2)^2 - (E_1 - E_2)^2}{p_1 p_2} = \varkappa^2(\varphi^2+\mu^2). \tag{13}$$

Thus we see that, whereas q_0^2/p_1p_2 and $|\vec{q}\,|^2/p_1p_2$ are of second order, their difference is of fourth order and can be neglected. We may write (from now on, q stands for $|\vec{q}\,|$)

$$q = q_0 = p_1 - p_2 = \varkappa p_1. \tag{14}$$

b) The following gauge transformation $\boldsymbol{\epsilon}' \to \boldsymbol{\epsilon} = \boldsymbol{\epsilon}' - \varepsilon'_0\boldsymbol{q}/q_0$ (see e.g. [7]) gives the unitary four-vector $\boldsymbol{\epsilon} = (0, 0, 1, 0)$ which is transverse to the direction $\vec{q}$.

One may thus identify the quantities $\boldsymbol{q}$ and $\boldsymbol{\epsilon}$ which we have just defined with the same quantities appearing in the cross-section for free photons (8), so that

$$\sigma_\mu(p_1) = \frac{e^2}{4\pi^2 c}\int (\mathrm{d}q/q^2)\,|\varepsilon_\nu \mathcal{T}^\nu|^2(2\pi)^4\,\delta^{(4)}\left(\sum \boldsymbol{p}_i + \boldsymbol{q} - \sum \boldsymbol{p}_f\right)\int\frac{\varphi^3\,\mathrm{d}\varphi}{(\varphi^2+\mu^2)^2} = \frac{e^2}{2\pi^2\hbar c}\int_{q_{\min}}^{q_{\max}} \sigma_{\mathrm{ph}}(q)\,\mathrm{d}q/q\int_0^{\varphi_{\max}}\frac{\varphi^3\,\mathrm{d}\varphi}{(\varphi^2+\mu^2)^2}. \tag{15}$$

$q_{\min}$ is given by the experiment ($100 \div 150$ MeV for stars in nuclear emulsions). We may take $q_{\max} \sim E_1/2$ because the calculations are meaningless for $q > E_2$. This region of validity of the method is largely sufficient for our purpose.

The integration over φ gives a log and is therefore not very sensitive to the upper limit. One may as well take $\varphi_{\max} \sim 1$, keeping in mind that φ will in general be small compared to unity and that our approximations become

(7) S. S. Schweber, H. A. Bethe and F. de Hoffmann: *Mesons and Fields*, vol. I (Evanston, 1955), p. 184 ff.

rather poor for $\varphi \gtrsim 1$. The convergence of the integral at the lower limit is insured automatically by the term μ^2. This is in contrast to the WW method, where a minimum impact parameter has to be rather arbitrarily introduced to make the result convergent. Thus finally

$$\sigma_\mu(E) = (2\alpha/\pi)[\ln(E/m) - \tfrac{1}{2}] \int \sigma_{\text{ph}}(q)\, \mathrm{d}q/q\,, \tag{16}$$

where $\alpha = e^2/4\pi\hbar c$ is the fine-structure constant and E stands for the initial energy of the μ-meson E_1. Comparison with (1) shows that Williams' and Weizsäcker's method gives in fact the same result as ours with only minor differences in the logarithmic factor.

3. – Conclusion.

We have thus shown that the application of the WW method to the present problem agrees with the field-theoretical point of view in the sense of equation (2) and the subsequent discussion. Needless to say that this method is valid for any electromagnetic interaction of a relativistic fermion.

George's interpretation of the nuclear interactions of μ-mesons seems therefore to be correct, as the predicted and experimental cross-sections are in fairly good agreement, and there is no need for the introduction of a new kind of interaction of the μ-meson with the nucleus. Note that in expression (3) the elastic Coulomb interaction is negligible as compared to the non-elastic photo-nuclear process (6).

We shall conclude with a few remarks about the angular deviation of the μ-meson in a nuclear interaction. The distribution of q has a maximum at $\sqrt{3}\mu$, and decreases slowly afterwards, so that the mean angle is about $\bar{\varphi} = \ln^{-1}(1/\mu) = \ln^{-1}(E/m)$. θ is related to φ by the relation $\theta = \varphi q/E$. Thus the deviation of the μ-meson for given primary energy E and energy transfer q has a rather extended distribution with mean value $\bar{\theta} = (q/E)\ln^{-1}(E/m)$. This is in contrast to the elastic case where θ is uniquely determined by q and E and is of the order q/E. For 15 GeV μ-mesons, $\bar{\theta}$ is thus almost one order of magnitude smaller than q/E.

In his work, George (1) has calculated the energy of the primaries of 1p stars by estimating the energy of the star and measuring the deviation of the primary, supposed to be a μ-meson, and by making use of the relation $\theta \sim q/E$. The agreement between the primariy spectrum found in this way and the known spectrum of the μ-mesons underground is thus probably fortuituous. The explanation is perhaps that the larger angles are produced by π-mesons or nucleons from higher energy nuclear interactions. For high energy

interactions where the primary is certainly a μ-meson, one of us (D.K.) has obtained some preliminary results with a large lead-plate cloud-chamber operated at 65 m w. e. underground to study penetrating showers produced by μ-mesons (at a rate of about one event per 10 days). The observed events provide in fact some indication that the angle of deviation of the μ-meson is much smaller than the estimated ratio of q/E, although in most cases it is of course almost impossible to identify the μ-meson unambiguously among the outgoing penetrating particles.

One should note that the small angular deviation of the μ-meson is already a consequence of the original WW theory: It is shown there that the electric and magnetic fields are mainly transverse to the path of the μ-meson and therefore the virtual photons proceed mainly along the direction of the meson and cannot be responsible for large transverse energy transfer.

* * *

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RIASSUNTO (*)

Dopo la discussione di argomenti avanzati contro l'applicazione del metodo semi-classico di Williams-Weizsäcker al problema delle interazioni nucleari di mesoni μ relativistici, si sviluppa un nuovo trattamento compatibile con la teoria quantistica dei campi. I risultati sono in accordo con quelli di Williams e Weizsäcker nel senso che le sezioni d'urto del mesone μ e del fotone sono confrontabili per processi corrispondenti per tramite di uno «spettro fotonico equivalente», quasi identico nelle due teorie. Si dimostra che nei processi fotonucleari anelastici la deviazione angolare del mesone μ non è unicamente in relazione coll'energia e il trasferimento d'energia primari. Nella maggior parte dei casi d'importanza pratica è molto minore che nello scattering coulombiano elastico.

(*) *Traduzione a cura della Redazione.*

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An Analysis of the Spin and Parity of the τ-Meson.

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Summary. — The generality of the double angular momentum expansion for the final state in τ-meson decay [1] is pointed out. The significance of the relativistic corrections is emphasized. A relativistic analysis is made of the energy and angular distribution of 481 collected events. One or more terms of the expansion are considered. In addition, a classical analysis is made of partially integrated distributions. On the basis of the analysis it is concluded that the τ-meson may have [spin, parity] of [0 —], [1+] or [3+] on the basis of a weak set of assumptions concerning the interaction. If stronger assumptions are utilized one can eliminate the [1+] case or even both the [1+] and [3+] cases.

1. – Introduction.

Fabri [1,2] has presented a method by which the most general distributions can be obtained for the pions resulting from the decay of a τ-meson of a given spin and parity. The method is relativistically correct if the relativistic kinematic relations between the variables are used.

The basis of the method is the expansion of the final state wave function in a double series of angular momentum eigenfunctions, which constitute a complete set for the problem. The usefulness of this method depends on the physical problem treated being representable by a small number of terms. In I it is demonstrated that this condition is fulfilled because of the short interaction distance.

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(1) E. Fabri: *Nuovo Cimento*, **2**, 479 (1954) (Hereafter referred to as I).

(2) R. H. Dalitz: *Phil. Mag.*, **44**, 1068 (1953)] *Phys. Rev.*, **94**, 1046 (1954); In this paper an expansion similar to that of Fabri is derived. It is applied to the classical limit.

It has been suggested [2] that the effects of pion-pion interaction should be treated separately, altering the results of the expansion. In fact, such interactions only have the possible effect of increasing the range of the interaction to that of the pion Compton wavelength. The series should still converge sufficiently rapidly (Sect. 5), probably necessitating at most the addition of one extra term. The number of parameters involved in stipulating a pion-pion interaction, in each isotopic spin state, of a certain range and strength is greater than the number of parameters inherent in an extra term of the Fabri expansion. In addition, a particular model must be chosen for the pion-pion interaction while the Fabri expansion is general. Also, it is extremely doubtful theoretically that the two pion interaction may be separated from the three pion interaction.

Similarly, effects of the Coulomb interaction are included in the expansion. Although the range of this interaction is large, the strength is weak, and high order terms should be negligible.

The method of I is specialized by certain assumptions about the coefficients relating to the unknown interaction. In Sects. 2 and 5 we discuss reasons for relaxing these restrictions. In Sect. 2 the results of I are summarized.

Throughout this work it will be stressed that, with the statistics now available, the relativistic corrections are of great importance. They alter some of the most important distributions by as much as 30 percent. In all previously published analyses of the energy and angular distribution of the odd pion [3,6], only classical expressions have been utilized. Some authors [1,4] have analyzed the data relativistically, but in a less complete way, masking some of the relativistic effects. In addition, the poorer statistics available to all the above authors made the relativistic corrections of less importance. In our analysis the relativistic corrections affect the result.

For this analysis all available published and unpublished data [3,4,6] have been collected, a total of 481 events. We are extremely grateful to those who made unpublished data available to us.

A test for bias previously formulated [5] has been corrected for relativistic

(3) B. T. Feld, A. C. Odian, D. M. Ritson and A. Wattenberg: *Phys. Rev.*, **100**, 1539 (1955); R. H. Dalitz: *Proceedings of the Rochester Conference* (1955); G. Costa and L. Taffara, *Nuovo Cimento*, **3**, 169 (1956), N. N. Biswas, L. Ceccarelli-Fabbrichesi, M. Ceccarelli, K. Gottstein, N. C. Varshneya and P. Waloschek: To be published; B. Bhowmik, D. Evans, I. J. van Heerden and D. J. Prowse: *Nuovo Cimento*, **3**, 574 (1956); K. P. Haddock: To be published; H. Winzeler, M. Teucher and E. Lohrmann: *Helv. Phys. Acta*, **29**, 75 (1956).

(4) E. Amaldi: *Proceedings of the Pisa Conference* (1955).

(5) Y. Eisenberg, S. Rosendorff and Y. Yeivin: *Nuovo Cimento*, **3**, 837 (1956).

(6) J. Orear, G. Harris and S. Taylor: *Phys. Rev.*, **102**, 1676 (1956).

effects, which were considerable. The total data satisfied this criterion well (the agreement with the classical expression is poor). However, in 65 of the events compilated by AMALDI [4] certain classes of events were omitted which in principle bias the energy distribution. We have not included these in the presented analysis of the energy distribution. In any case, due to the small number of cases involved, their exclusion alters none of our results.

In Sects. **3** and **4** we consider the totally integrated energy and angular distributions of the odd pion, with relation to the leading term of the expansion only. This analysis isolates τ-mesons of [spin, parity] $[0-]$ or $[3+]$ as the only cases of spin < 4 satisfying such distributions with reasonable probability.

In Sect. **5** the effect of the next term of the expansion is considered for the cases $[0-]$ and $[1+]$. No improvement is obtainable for the $[0-]$ case. The $[1+]$ case however is made much more likely by the addition of this term.

In Sect. **6** we consider a more detailed analysis of the data in order to obtain further distinctions between the cases which are acceptable in the sense of Sects. 3-5. Only partially integrated energy and angular distributions are utilized. Insufficient data prevent a distinction from being made by this means, but the position of the $[1+]$ possibility is greatly strengthened.

In Sect. **7** a lifetime argument eliminating τ-meson spins $\gtrsim 4$ is discussed. Also, in that section the results for lower spins are summarized.

2. – The Expansion of Fabri.

Designating $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ as the co-ordinates and momenta of the like pions in the τ-meson system, and $\boldsymbol{r}_3$ and $\boldsymbol{p}_3$ as the same variables of the unlike pion, FABRI performs the following canonical transformation.

$$(1)\qquad \left\{\begin{array}{ll} \boldsymbol{R} = \dfrac{1}{\sqrt{3}}(\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3) & \boldsymbol{P} = \dfrac{1}{\sqrt{3}}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)\,, \\[2ex] \boldsymbol{r} = \dfrac{1}{\sqrt{2}}(\boldsymbol{r}_2 - \boldsymbol{r}_1) & \boldsymbol{p} = \dfrac{1}{\sqrt{2}}(\boldsymbol{p}_2 - \boldsymbol{p}_1)\,, \\[2ex] \boldsymbol{r}' = \sqrt{\dfrac{2}{3}}\left[\boldsymbol{r}_3 - \dfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)\right] & \boldsymbol{p}' = \sqrt{\dfrac{2}{3}}\left[\boldsymbol{p}_3 - \dfrac{1}{2}(\boldsymbol{p}_1 + \boldsymbol{p}_2)\right], \end{array}\right.$$

and also defines

$\boldsymbol{k} = \boldsymbol{r}\times\boldsymbol{p}$ the relative angular momentum of the like pions, and

$\boldsymbol{k}' = \boldsymbol{r}'\times\boldsymbol{p}'$ the angular momentum of the unlike pion relative to the center of mass of the like pions.

As the τ-meson decays at rest we have $\boldsymbol{P} = 0$, leaving only $\boldsymbol{p}$ and $\boldsymbol{p}'$ as independent momenta. Thus, in a momentum representation, one may expand the asymptotic final state wave function in terms of a complete set of solutions for two free particles, in eigenfunctions of k and k'.

In the eigenfunctions the spherical Bessel functions $j^l(pr)$ are approximated by $[(2l+1!!)]^{-1}(pr)^l$. This is a good approximation when $(pr)^2 \ll (4l+6)$, a condition which is fulfilled by all relevant pr. For a τ-meson of spin J and parity P, we then have the probability distribution in terms of p' and θ (θ is the angle between $\boldsymbol{p}'$ and $\boldsymbol{p}$)

$$(2)\quad \begin{cases} F_{JP}(p', \theta) = A_{JP} \sum\limits_{j} \left| \sum\limits_{ll'} \alpha_{ll'} \left(\dfrac{m}{M}\right)^{l+l'} X_{ll'j} \left(\dfrac{p}{m}\right)^{l} \cdot \left(\dfrac{p'}{m}\right)^{l'} P_{l'}^{|j|}(\cos\theta) \right|^2 , \\[2ex] X_{ll'j} = (ll'0j|ll'Jj) \left\{ (2l+1)(2l'+1) \dfrac{(l - |j|)!}{(l' + |j|)!} \right\}^{\frac{1}{2}} \cdot \\[2ex] \qquad \cdot \dfrac{1}{(l+l'+4)!!} \cdot \dfrac{(l+1)!!}{(2l+1)!!} \cdot \dfrac{(l'+1)!!}{(2l'+1)!!} . \end{cases}$$

j is the magnetic quantum number related to J. l and l' are the quantum numbers of k and k' respectively. A_{JP} depends only on J and P. M and m are the τ-meson and pion masses respectively.

Under the following assumptions a) and b) stipulated in I, $\alpha_{ll'} = \alpha_{l+l'}$, $|\alpha_{l+l'}| \lesssim 1$ and decreases with increasing $\lambda = l + l'$.

a) The interaction radius is the τ-meson Compton wavelength.

b) The interaction is independent of l and l'.

Both assumptions are not necessary physical conditions. We shall discuss their validity and the effect of ignoring them in Sect. **5**. However, if a) and b) hold, then, as remarked in I, only those terms of least λ are of importance. The terms of least λ are more than a factor 10^2 greater than the terms of higher order.

It is to be noted that in « degenerate » cases, those in which there is more than one term of least λ, the results of the analysis are very sensitive to assumption b). Assumption b) infers that $\alpha_{ll'} = \alpha_{\lambda}$. This assumption is not likely to be true to the extent of prohibiting magnitude variations of order unity or phase differences between $\alpha_{ll'}$ of equal λ.

It has been suggested [6] that the principle of time reversal limits the phase difference to 0 or π. This conclusion, however, only applies in the perturbation limit or for the weighting function in the time reversible eigenwave function representation [7]. Our wave function, representing incoming τ-mesons and

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[7] L. C. BIEDENHARN and M. E. ROSE: *Rev. Mod. Phys.*, **25**, 729 (1953).

outgoing pions, is not time reversible. The coefficients $\alpha_{ll'}$ are in fact not generally of equal phase as is evident from equation (27) of reference ([7]) (the $\alpha_{ll'}$ are proportional to the S function of this reference). In our analysis we shall consider such phase and magnitude difference, with important effect on the results.

If we denote the kinetic energy of the odd pion in the τ-meson system by K, the probability distribution in terms of K and θ only, $F_{JP}(K,\theta)$, may be found by using the relations ($c = 1$)

$$p'^2 = \tfrac{3}{2}K(K + 2m)$$

and

$$p^2 = M(K_m - K)\left[1 + \frac{K(K+2m)\cos^2\theta}{(M-m-K)^2 - K(K+2m)\cos^2\theta}\right], \tag{3}$$

K_m (the maximum value of K) $= (M+m/2M)Q$, where Q is the kinetic energy released during the disintegration.

Equation (3) illustrates the order of the relativistic correction. This correction alters the energy dependence of p' by $O(K/2m) \lesssim 0.2$. The relativistic term in p yields a combined energy and angular dependence of ~ 20 percent. The angular dependence is entirely relativistic. The degree to which these corrections affect the terms of equation (2) increases quickly with l.

To obtain an integrated probability it is necessary to multiply (2) by the statistical weight

$$\begin{aligned} S(K,\theta)\,\mathrm{d}K\,\mathrm{d}\cos\theta \propto (K+m)&\left[\frac{2MK(K_m-K)(K+2m)}{(M-m-K)^2}\right]^{\frac{1}{2}}\cdot\\ &\cdot\left[(M-m-K)^2 - \frac{2MK(K_m-K)(K+2m)\cos^2\theta}{(M-m-K)^2 - K(K+2m)\cos^2\theta}\right]\cdot\\ &\cdot\left[1-\frac{K(K+2m)\cos^2\theta}{(m-m-K)^2}\right]^{-\frac{3}{2}}\mathrm{d}\cos\theta\,. \end{aligned} \tag{4}$$

Here again the relativistic correction is large (~ 30 percent for large K) although the angular dependent part is only ~ 10 percent in this case. The larger part of this correction is given in I by the distortion of the boundary in Fig. 1, which is classically a circle. The above remarks together with the explicit calculations of Sects. **3-5** indicate clearly that the neglect of relativistic terms is not justified in view of the presently available statistics. This is most important with regard to the energy distribution of the odd pion.

3. - The $\cos\theta$ Distribution and its Relativistic Corrections.

3·1. $\cos\theta$ *from experiment.* – In most cases either the kinetic energies of all three pions are known, or two are known and the third can be calculated from the Q value. The value of $\cos\theta$ is then given classically by the following simple relation

$$\cos\theta_c = \frac{(K_1 - K_2)}{\sqrt{3K(K_m^c - K)}}\,, \tag{5}$$

where K_1 and K_2 are the kinetic energies of the faster and slower like pions respectively, and K_m^c is the classical value of the maximum K obtainable.

$$K_m^c = \frac{2Q}{3}\,. \tag{6}$$

Relativistically the formula equivalent to (5) is

$$\cos\theta = \frac{(K_1 - K_2)(Q + 2m - K)}{\sqrt{K(K + 2m)[2M(K_m - K) + (K_1 - K_2)^2]}}\,, \tag{7}$$

where K_m is the true maximum value of K used in equation (3). It is 4 percent smaller than K_m^c.

It is important to use (7) rather than (5) because, for the majority of (K_1, K_2, K), $\cos\theta$ is more than 10 percent larger than $\cos\theta_c$. This one sided effect of the relativistic correction can be understood by noting that for $K < K_m$ we have $(K/2) < Q - K$ and thus

$$\frac{(Q + 2m - K)}{\sqrt{(K + 2m)2m}} > \frac{2m + (K/2)}{\sqrt{(K + 2m)2m}} \approx 1\,,$$

while for $K \approx K_m$ we have $(M/3m)(K_m - K) \ll K_m^c - K$ tending to make $\cos\theta$ much greater than $\cos\theta_c$. The term proportional to $(K_1 - K_2)^2$ in the radical is usually negligible as $\lim_{K\to K_m}(K_1 - K_2) = 0$. However, in extreme cases it is possible for this term to compensate for the difference $(K_m^c - K_m)$. In all the events analyzed we obtained $\cos\theta > \cos\theta_c$.

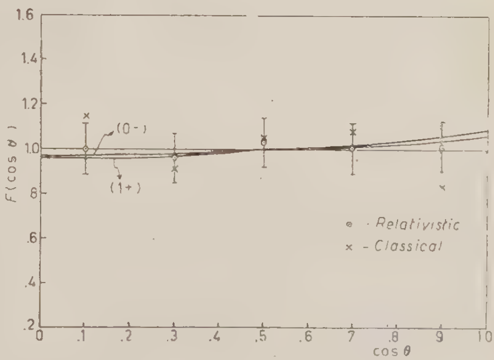

Fig. 1. – Relativistic angular distribution of the first term of the $[0-]$ and $[1+]$ expansion. The experimental data are shown classically as well as relativistically.

In Fig. 1 $\cos\theta_c$ (the quantity used in all previous analyses[3]) and $\cos\theta$ are plotted for all collected events. It is evident that $\cos\theta$ is much more isotropic than $\cos\theta_c$. The effect of using the proper relativistic expression is considerable.

For the cases in which less than two mesons stopped in the emulsion, the following relativistic formula determines K_i $(K - K_3)$ in terms of α_i the angle opposite K_i in the τ-meson system

$$K_i = Q - \left[\sqrt{K_i(K_i + 2m)\frac{\sin^2\alpha_k}{\sin^2\alpha_i} + m^2} - m\right] - \tag{8}$$
$$- \left[\sqrt{K_i(K_i + 2m)\frac{\sin^2\alpha_j}{\sin^2\alpha_i} + m^2} - m\right].$$

One can then substitute the energies obtained into equation (7). Due to the small number of cases in this class the classical approximation to (8) may be used with insignificant effect on the histogram. Substituting the classical approximation of (8) into (5) we have

$$\cos\theta_c = \frac{(\sin^2\alpha_1 - \sin^2\alpha_2)}{\sin\alpha_3\sqrt{2\sin^2\alpha_1 + 2\sin^2\alpha_2 - \sin^2\alpha_3}}, \tag{9}$$

In those cases in which the odd pion is not known there are two possible values of $\cos\theta$. We then consider the two possible distributions in which the smaller or larger values of $\cos\theta$ are chosen. The most probable distribution is the mean with a statistical deviation of the order of $\sqrt{n}/2$ (n the number of doubtful cases in a region of the histogram) to be added to the counting statistical deviation $\sqrt{N}$ (N the total number of cases in that region of the histogram).

3·2. *Theoretical* $\cos\theta$ *distribution, least* λ *only.* – To obtain the fully integrated $\cos\theta$ distribution (2) is multiplied by (4) and the integral over $\mathrm{d}K$, $0 < K \leqslant K_m$, is performed. The normalized (all numerical F's are normalized to unity) relativistic result for the least λ term of the [0—] case is

$$F_{0-}(\cos\theta) = \int_0^{K_m}\mathrm{d}K S(K,\theta) = 0.968\,[1 + 0.098\cos^2\theta]\,. \tag{10}$$

For the [1+] case the result is

$$F_{1+}(\cos\theta) \propto \int_0^{K_m}\mathrm{d}K K(K + 2m)S(K,\theta) \propto 0.958[1 + 0.130\cos^2\theta]\,. \tag{11}$$

The 10 percent anisotropy in each case is an entirely relativistic effect. The results are plotted together with the experimental data in Fig. 1. The agreement is seen to be excellent. A χ^2 test yield 98.8 percent for the [0 −] and 98.0 percent for the [1+] case.

The [even +] and [odd −] cases (those in which two pion decay is possible) [1 −], [2+], and [3 −] are inconsistent with the experimental angular distribution. We need only remark that in all these cases the distribution is proportional to $(1 - \cos^2\theta)$ relativistically as well as classically. As the data do not decrease for $\cos\theta > \frac{1}{2}$, with only a small statistical error, these cases are very improbable. In the [1 −] case the distribution also contains a factor $\cos\theta$, and the same argument may be made with regard to $\cos\theta < -\frac{1}{2}$. In Sect. **4** it is shown that the energy distribution of these cases is also inadequate.

The [2 −] and [3+] cases are of more interest. These are « degenerate » and thus, according to our discussion at the end of Sect. **2**, we may vary the ratio of the relevant α_{ij} providing the magnitude of this quantity is of order unity.

In Sect. **4** we shall demonstrate that [2 −] is very unlikely on grounds of the energy distribution only. For that reason we will rely on the classical angular distribution to give us an indication of the actual angular distribution. We have

$$(12)\qquad F_{2-}(\cos\theta) \propto \int_0^{K_m} \mathrm{d}K S(K,\theta)\left[|\alpha_{02}|^2\left(\frac{p'}{m}\right)^4 + |\alpha_{20}|^2\left(\frac{p}{m}\right)^4 + \right.$$

$$\left. + \left(\frac{p'}{m}\right)^2\left(\frac{p}{m}\right)^2 |\alpha_{02}|\,|\alpha_{20}|\cos\varphi(3\cos^2\theta - 1)\right] \propto$$

$$\propto \left[1 + \left(\frac{M}{3m}\right)^2 \left|\frac{\alpha_{20}}{\alpha_{02}}\right|^2\right]^{-1}\left[1 + \left(\frac{M}{3m}\right)^2 \left|\frac{\alpha_{20}}{\alpha_{02}}\right|^2 + \frac{3}{5}\left(\frac{M}{3m}\right)\left|\frac{\alpha_{20}}{\alpha_{02}}\right|(3\cos^2\theta - 1)\cos\varphi\right].$$

It is seen that the classical term is isotropic when $\cos\varphi$ (φ = phase $\alpha_{02}\,\alpha_{20}$) vanishes. A slightly different $\cos\varphi$ will make (12), including the relativistic terms, almost isotropic.

Contrary to the result obtained by applying assumption *b*) of Sect. **2** strictly, [2 −] is satisfactory for the angular distribution.

The relativistic result for [3+] is

$$F_{3+}(\cos\theta) \propto \int_0^{K_m} \mathrm{d}K S(K,\theta)\left[16|\alpha_{03}|^2\left(\frac{p'}{m}\right)^6 + 21|\alpha_{21}|^2(2+\cos^2\theta)\left(\frac{p}{m}\right)^4\left(\frac{p'}{m}\right)^2 + \right.$$

$$\left. + 12\sqrt{7}\cos\varphi|\alpha_{03}|\,|\alpha_{21}|(3\cos^2\theta - 1)\left(\frac{p}{m}\right)^2\left(\frac{p'}{m}\right)^4\right].$$

Therefore

$$(13)\qquad F_{3+}(\cos\theta) = \left[1.83 + 2.39\left|\frac{\alpha_{21}}{\alpha_{03}}\right|^2 + 0.07\cos\varphi\left|\frac{\alpha_{21}}{\alpha_{03}}\right|\right]^{-1}\cdot$$

$$\cdot\left[(1.74 + 0.28\cos^2\theta) + \left|\frac{\alpha_{21}}{\alpha_{03}}\right|^2(0.94 + 0.18\cos^2\theta)(2+\cos^2\theta) + \right.$$

$$\left. + \cos\varphi\left|\frac{\alpha_{21}}{\alpha_{03}}\right|(1.41 + 0.29\cos^2\theta)(3\cos^2\theta - 1)\right].$$

It is not possible to attain exact isotropy in this case, but for $|\alpha_{21}/\alpha_{03}| = 1.43$, $\cos\varphi = -0.5$ there is a variation of only 2 percent in the distribution (see Fig. 2). The energy distribution for this case is plausible for the appropriate $|\alpha_{03}/\alpha_{21}|$ as will be shown in Sect. **4**. Thus, this case is very interesting. For the angular distribution the χ^2 test yields 99 percent.

In Sect. **5** we shall discuss the case of $[1+]$ with the next order term in λ. This case is very similar to that of the least λ $[3+]$ terms. The similarity arises from the fact that in the former case (l, l') is (0, 1) and (2, 1) while in the latter it is (0, 3) and again (2, 1) yielding a very similar structure for $F(K, \theta)$. All higher spins for the τ-meson lead to « degeneracy ». It will be possible in some of these cases to obtain approximate isotropy by adjusting the $\alpha_{ll'}$ of least λ. We shall discuss spins > 3 in Sect. **7**.

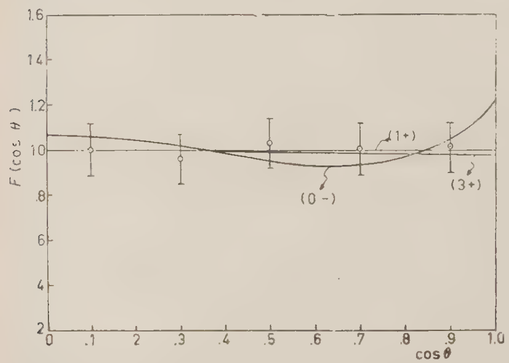

Fig. 2. – Relativistic angular distributions for the first two terms of the $[0-]$, $[1+]$ and $[3+]$ expansions. For $[0-]$, $|\alpha_{22}/1200\alpha_{00}| = 1$ and $\cos\varphi = -0.15$; for $[1+]$, $|\alpha_{21}/35\alpha_{01}| = 1.73$ and $\cos\varphi = 0.45$; and for $[3+]$, $|\alpha_{21}/\alpha_{03}| = 1.43$ and $\cos\varphi = -0.5$.

4. – Relativistic K Distributions.

4·1. *Experimental K distribution.* – When the odd pion has stopped in the emulsion, the value of K obtained from the range energy relation is used. If the two like pions, but not the unlike pion, stop in the emulsion, K is obtained from the Q value (75 MeV). In the few remaining events one can obtain K from the angles α_1, α_2, α_3 by using the relativistic equation (8). Due to the small errors and small number of events involved the classical

approximation

$$K = \frac{3}{2}\frac{\sin^2\alpha_3}{(\sin^2\alpha_1 + \sin^2\alpha_2 + \sin^2\alpha_3)} K_m \, . \tag{14}$$

is used for these cases.

In presenting the data it is useful to normalize to the statistical weight, thus eliminating a dependence common to all distributions. We must replace the classical statistical weight $S^c(K)$ by the integrated relativistic weight

$$S(K) = \int_0^1 S(K,\theta)\,\mathrm{d}\cos\theta = 5.2\,\frac{(K+m)}{m^{\frac{9}{2}}}\left[\frac{K(K+2m)(K_m-K)}{(M-m)^2-2MK}\right]^{\frac{1}{2}}\cdot \left[(M-m-K)^2-\frac{2MK(K+2m)(K_m-K)}{3[(M-m)^2-2MK]}\right]. \tag{15}$$

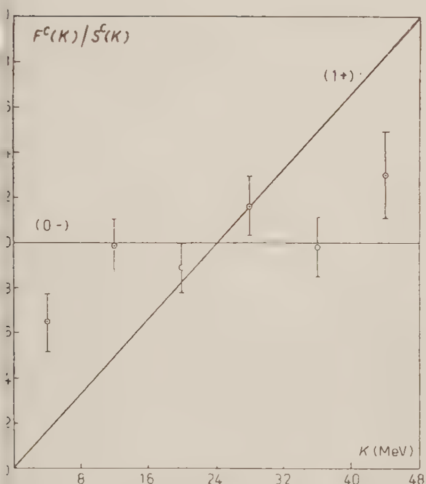

Fig. 3. – The classical energy distribution of the odd pion, normalized to the classical statistical weight. The [0 —] and [1+] cases with first term only are plotted.

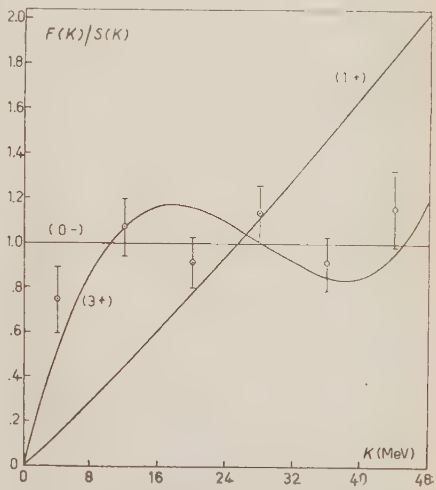

Fig. 4. The relativistic energy distribution of the odd pion, normalized to the relativistic statistical weight. The least λ terms of [0 —], [1+] and [3+] cases are plotted. The [3+] coefficients are as in Fig. 2.

The distributions, normalized by $S^c(K)$ and $S(K)$ are plotted in Fig. 3 and Fig. 4, respectively.

The classical result has been plotted on the assumption that $K^c_m = 48$ MeV. This is done to avoid the artificially small contribution to the last region of

the histogram which is obtained when K_m is chosen to include a region where no events can occur. This is, in fact a relativistic correction, but one not based on a detailed knowledge of the relativistic terms of the distribution. The true value of the maximum energy is based only on a direct kinematic deduction. Thus we believe that this relativistic effect should be taken into account even when ignorance of the details of the distribution is assumed.

As K is usually obtained directly, and is thus relativistically correct, our experimental distribution does not differ from that which a classical analysis would have obtained with the same data. This contrasts with the situation for the experimental angular distribution.

4·2. *Theoretical K distributions, least λ only.* – The result for [0 —] is just the integrated statistical weight $S(K)$ given by equation (15). This plots in Fig. 4 as a straight line, and deviates from the classical result by as much as 30 percent. The difference in normalization between Fig. 3 and 4 eliminates the difference between the classical and relativistic [0 —] curves; but the effect of the relativistic correction can be seen in the different relation of the data to the theoretical curves in these two figures. The χ^2 test yields 20 percent for the relativistic distribution and 2 percent for the classical

For [1+] we have

$$F_{1+}(K) = \frac{2.47}{m^2} K(K + 2m)S(K) . \tag{16}$$

This distribution is plotted in Fig. 4, and its classical approximation in Fig. 3. The correspondence to the data is seen to be poorer for the relativistic than for the classical curve. The χ^2 tests yields $\ll 0.1$ percent for the relativistic and $\ll 0.1$ percent for the classical distributions.

For the [odd —] and [even +] cases neither l nor l' can be zero for any term of the expression. Thus, their energy distributions vanish at $K = K_m$ as well as at $K = 0$ after normalizing by the statistical weight. Due to the fact that the number of events in the last 8 MeV interval is greater than that in any other interval, the [1 —], [2+] and [3 —] least λ types do not correspond to the data. The agreement is much worse than in the [1+] or [3+] in which only the first interval, in which the number of points is indeed small, is affected. It is possible for very high order terms or very high spins, that the drop to zero takes place in a very small interval, and thus will not be apparent in the 8 MeV interval. But for the first two terms of the spins considered here, the decrease is great over the whole 8 MeV. Together with the evidence in Sect. 3 concerning the angular distribution, it is apparent that these « two pion » decay cases do not correspond to the τ-meson.

[2 —] is a « degenerate » case with least λ $(l, l') = (0, 2)$ and $(2, 0)$. In

this case we have

$$(17)\qquad F_{2-}(K) \propto \frac{S(K)}{m^4}\Bigg[K^2(K+2m)^2 + \frac{4}{9}\left|\frac{\alpha_{20}}{\alpha_{02}}\right|^2 M^2(K_m - K)^2 A(K) + \frac{4}{9}\cos\varphi\left|\frac{\alpha_{20}}{\alpha_{02}}\right| MK(K+2m)(K_m - K)(\gamma^2 - 1)B(K)\Bigg],$$

where $\gamma^2 = (M - m - K)^2[(M-m)^2 - 2MK]^{-1}$, and where $A(K)$ and $B(K)$ are slowly varying functions of K,

$$A(K) = \left[(M - m - K)^2 + \frac{2M(K_m - K)(1-\gamma^2)}{3}\right]^{-1}\left[\left(\frac{1}{5}\gamma^4 - \frac{4}{15}\gamma^2 - \frac{8}{15}\right)\cdot (M-m-K)^2 + 2M(K_m - K)\left(\frac{1}{7}\gamma^4 + \frac{4}{35}\gamma^2 + \frac{8}{105}\right)(1-\gamma^2)\right],$$

$$B(K) = \left[(M - m - K)^2 + \frac{2M(K_m - K)(1-\gamma^2)}{3}\right]^{-1}\cdot\left[(M-m-K)^2 - \frac{2}{5}M(K_m - K)(3\gamma^2 - 1)\right].$$

In Sect. 3 we demonstrated that the $[2-]$ could fit the angular distribution. Despite the freedom given by the «degeneracy» it is not possible to match the K distribution. This can be seen as follows. The classical approximation to (17) is

$$(18)\qquad \frac{F^c_{2-}(K)}{S^c(K)} = \frac{16\left[K^2 + \left|\frac{\alpha_{20}}{\alpha_{02}}\right|^2\left(\frac{M}{3m}\right)^2(K_m - K)^2\right]}{5(K^c_m)^3\left(1 + \left|\frac{\alpha_{20}}{\alpha_{02}}\right|^2\left(\frac{M}{3m}\right)^2\right)},$$

which is a parabola. The minimum value of $(F^c/S^c)_{\max}/(F^c/S^c)_{\min}$ is 3, and occurs when K is at $K_m/2$. This is in contradiction to the data for which

$$\frac{F/S(0)}{F/S(K_m/2)} \approx 0.75 \qquad \text{and} \qquad \frac{F/S(K_m)}{F/S(K_m/2)} \approx 1.55\,.$$

Shifting the minimum to large K makes the agreement with the first experimental ratio worse; while shifting the minimum to small K makes the agreement with the second curve worse. The relativistic terms of (17) make $F(K)$ increase more quickly with K thus again diminishing the agreement with the second ratio. Relativistic effects are negligible for $K < K_m/2$, thus the improvement obtained with respect to the first inequality is also negligible.

Thus, some experimental points will differ from the theoretical distribution by more than a factor of 3, and the correspondence will be extremely poor.

For the [3+] case $(l, l') = (2, 1)$ and $(0, 3)$ leading to

$$\frac{F_{3+}(K)}{S(K)} \propto \frac{K^3(K+2m)^3}{m^6} + \frac{49}{36}\left|\frac{\alpha_{21}}{\alpha_{03}}\right|^2 \frac{M^2K(K+2m)(K_m-K)^2}{m^6}\,C(K) + \\ + \frac{\sqrt{7}}{2}\cos\varphi\left|\frac{\alpha_{21}}{\alpha_{03}}\right| \frac{MK^2(K+2m)^2(K_m-K)}{m^6}(\gamma^2-1)D(K)\,, \tag{19}$$

where $C(K)$ and $D(K)$ are slowly varying functions of K

$$C(K) = \left[(M-m-K)^2 - \frac{2M(K_m-K)}{3}(\gamma^2-1)\right]^{-1}\left[\left(\frac{9}{35}\gamma^4 + \frac{2}{7}\gamma^2 + \frac{16}{15}\right)\cdot \\ \cdot(M-m-K)^2 - \frac{6M(K_m-K)(\gamma^2-1)}{49}\left(3\gamma^4 + 2\gamma^2 + \frac{16}{35}\right)\right],$$

$$D(K) = \left[(M-m-K)^2 - \frac{2M}{3}(K_m-K)(\gamma^2-1)\right]^{-1}\cdot \\ \cdot\left[\frac{2}{3}(M-m-K)^2 - \frac{4M}{5}(K_m-K)\left(\gamma^2-\frac{1}{3}\right)\right].$$

Reasonable agreement with experiment can be obtained for $\cos\varphi = -0.5(\pm 0.2)$ and $|\alpha_{03}/\alpha_{21}| = 1.43(\pm 0.1)$.

These parameters are consistent with the fit obtained to the angular distribution. The curve is plotted in Fig. 4. The χ^2 test yields 3 percent.

5. – Inclusion of Terms of Higher Order in λ.

Assumption *a*) of Sect. 2 implies that the field quanta responsible for the range are the τ-mesons. This is in fact not known while the theoretical interaction Hamiltonian remains unknown. In fact, any interaction distance between the nucleon Compton wavelength (if the interaction is caused by the exchange of nucleon pairs) and the pion Compton wavelength seems to be equally likely at the present stage of our knowledge. In the latter case succeeding terms in the expansion would not decrease so rapidly as indicated in I. In equation (2) the $\alpha_{ll'}$ would retain their properties only if the appropriate interaction distance was substituted for the τ-meson Compton wavelength represented by M. However, the series still decreases rapidly if assumption *b*) applies. If the interaction distance is the pion Compton wave length and *b*) applies we require $|\alpha_{21}/\alpha_{01}| \approx 35$ in order to give equal weight to the first and

second term of the [1 +] case. This is the minimum ratio encountered among the cases considered.

However, assumption *b*) is also very doubtful physically. The interaction may well be l and l' dependent. It is not unlikely that absolute or partial selection rules exist for some values of l and l'. If this is the case the series may decrease rapidly over a period of several terms, but certain terms may be abnormally suppressed. In particular the first term may be suppressed so that it no longer dominates.

The situation may be compared to that in nuclear or atomic radiation theory where a perturbation expansion in the radiation field is assumed. If the transition probability for a certain case is much smaller than that usual for the lowest order radiation, it is assumed that a selection rule applies to this order. In such a case the next order term is compared with experiment. If its magnitude and angular dependence is correct the interpretation is accepted and all higher order terms neglected. If only a partial selection rule is operative for the first term, a mixture of the first and second term may have to be considered. The difference in the present case is that little is known about the absolute magnitude, and only the shape of the distribution may be used as a guide.

We will consider the effects of including the second order terms in λ in the [0 —] and [1 +] cases. As has already been noted, the situation is not sufficiently improved by such a consideration in the « two pion decay » cases. The [2 —] and [3 +] cases are also not treated in this way because of the large number of free parameters which would be involved. It may be worthwhile to consider [2 —] and [3 +] with second order terms if the evidence should in the future exclude the lower spins, or [3 +] with lowest order terms.

For the [0 —] case we have

$$F_{0-}(K,\theta) \propto S(K,\theta)\left[1+\frac{45M^2}{16m^8}\left|\frac{\alpha_{22}}{1200\,\alpha_{00}}\right|^2 K^2(K+2m)^2(K_m-K)^2\cdot\right.$$
$$\cdot\left(\frac{3\cos^2\theta-1}{\sin^2\theta+(\cos\theta^2)/\gamma^2}\right)^2+\frac{3\sqrt{5}M}{2m^4}\left|\frac{\alpha_{22}}{1200\,\alpha_{00}}\right|\cdot$$
$$\left.\cdot\cos\varphi\,K(K+2m)(K_m-K)\left(\frac{3\cos^2\theta-1}{\sin^2\theta+(\cos^2\theta)/\gamma^2}\right)\right], \tag{20}$$

leading to the angular distribution

$$F_{0-}(\cos\theta)=\left[1.18+0.094\left|\frac{\alpha_{22}}{1200\,\alpha_{00}}\right|^2+0.0339\cos\varphi\left|\frac{\alpha_{22}}{1200\,\alpha_{00}}\right|\right]^{-1}\cdot$$
$$\cdot\left[(1.14+0.112\cos^2\theta)+0.0777\left|\frac{\alpha_{22}}{1200\,\alpha_{00}}\right|^2(3\cos^2\theta-1)^2(1.35+0.32\cos^2\theta)+\right.$$
$$\left.+0.530\cos\varphi\left|\frac{\alpha_{22}}{1200\,\alpha_{00}}\right|(3\cos^2\theta-1)(1.24+0.21\cos^2\theta)\right]+O\left(\frac{1}{10}\,\frac{K_m^2}{m^2}\right), \tag{21}$$

and the energy distribution

$$(22)\qquad \frac{F_{0-}(K)}{S(K)} \propto 1 + \frac{9M^2}{4m^8}\left|\frac{\alpha_{22}}{1\,200\,\alpha_{00}}\right|^2 K^2(K+2m)^2(K_m-K)^2E(K) +$$

$$+\frac{\sqrt{5}M}{m^4}\cos\varphi\left|\frac{\alpha_{22}}{1\,200\,\alpha_{00}}\right| K(K+2m)(K_m-K)(\gamma^2-1)B(K)\,,$$

where $B(K)$ is defined for equation (17), and

$$E(K) = \left[(M-m-K)^2 - \frac{2M}{3}(K_m-K)(\gamma^2-1)\right]^{-1}\cdot$$

$$\cdot\left[\left(\gamma^4 - \frac{2}{3}\gamma^2 + \frac{2}{3}\right)(M-m-K)^2 - \frac{10M}{7}(K_m-K)(\gamma^2-1)\left(\gamma^4 - \frac{2}{5}\gamma^2 + \frac{2}{15}\right)\right].$$

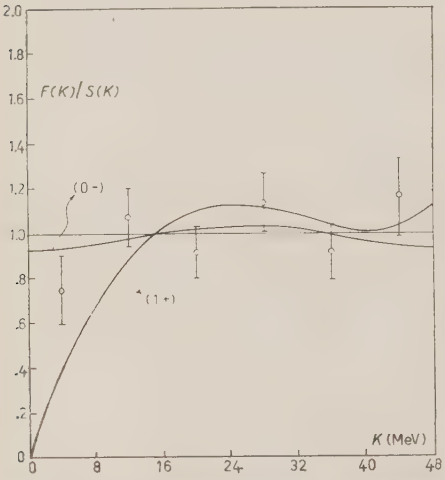

Fig. 5. - The relativistic energy distribution of the odd pion, normalized to the relativistic statistical weight. The first two terms in the expansion in λ are plotted for the [0 —] and [1+] cases. The ratios of the coefficients are as in Fig. 2.

We note that the energy distribution is approximately symmetric for all values of α_{22}/α_{00} and $F/S(K_m) = F/S(0)$. The contribution to the last interval of the histogram is thus only a little larger than to the first interval for all values of $|\alpha_{22}/\alpha_{00}|$ which do not make $F(K_m/2)/F(0)$ too large. In increasing $|\alpha_{22}/\alpha_{00}|$ from zero, the advantages and disadvantages with regard to fitting the experimental points cancel each other until $|\alpha_{22}/1200\alpha_{00}| \approx 1$. The fit worsens beyond this and becomes poor for $|\alpha_{22}/1200\alpha_{00}| > 1.5$. The energy distribution is insensitive to $\cos\varphi$ as $\cos\varphi$ is the coefficient of a relativistic term. For $|\alpha_{22}/1\,200\,\alpha_{00}| = 1$ and $\cos\varphi = -0.15$ the χ^2 test yields 20 percent. The curve is reproduced in Fig. 5.

We see from equation (21) that for $\cos\theta \gtrsim 0$ the $\cos^4\theta$ term is of opposite sign to the $\cos^2\theta$ term. Thus some cancellation is obtained. For $|\alpha_{22}/1200\alpha_{00}| < 1$ this cancellation is just sufficient to keep the anisotropy within the experimental statistical deviations. For $|\alpha_{22}/1200\alpha_{00}| = 1$ and $\cos\varphi = -0.15$ the χ^2 test yields 84 percent. $\cos\varphi$ may change by no more than ± 0.05 without

making the agreement very poor. The curve for the above case is plotted in Fig. 2.

Although no advantage is gained by the addition of the second order term, a substantial contribution from this term cannot be ruled out.

For a [1+] τ-meson the addition of a second order term is more fruitful.

$$(23)\qquad F_{1+}(K,\theta) \propto S(K,\theta)\,\frac{K(K+2m)}{m^2}\cdot$$
$$\cdot\left[1+\frac{M^2}{2m^4}\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|^2 (K_m-K)^2\,\frac{(1+3\cos^2\theta)}{(\sin^2\theta+(\cos^2\theta)/\gamma^2)^2}+\right.$$
$$\left.+\frac{\sqrt{2}M}{m^2}\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|\cos\varphi(K_m-K)\,\frac{(1-3\cos^2\theta)}{(\sin^2\theta+(\cos^2\theta)/\gamma^2)}\right].$$

The angular distribution is

$$(24)\qquad F_{1+}(\cos\theta)=\left[1.35+0.367\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|^2-0.0345\cos\varphi\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|\right]^{-1}\cdot$$
$$\cdot\left[(1.29+0.168\cos^2\theta)+0.139\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|^2(1+3\cos^2\theta)(1.19+0.228\cos^2\theta)+\right.$$
$$\left.+0.645\cos\varphi\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|(1-3\cos^2\theta)(1.234+0.207\cos^2\theta)\right]+O\left(\frac{1}{10}\,\frac{K_m^2}{m^2}\right),$$

and the energy distribution is

$$(25)\qquad \frac{F_{1+}(K)}{S(K)}\propto\frac{K(K+2m)}{m^2}\cdot$$
$$\cdot\left[1-\frac{M^2}{m^4}\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|^2(K_m\quad K)^2G(K)\quad\frac{2\sqrt{2}}{3}\,\frac{M}{m^2}\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|\cos\varphi(K_m-K)\cdot(\gamma^2-1)B(K)\right],$$

where $B(K)$ is defined for equation (17), and

$$G(K)=\left[(M-m-K)^2-\frac{2M}{3}(K_m-K)(\gamma^2-1)\right]^{-1}\cdot$$
$$\cdot\left[\left(\frac{2\gamma^4}{5}+\frac{1}{3}\gamma^2+\frac{4}{15}\right)(M-m\quad K)^2\quad 2M(K_m-K)(\gamma^2\quad 1)\left(\frac{2}{7}\gamma^4+\frac{1}{7}\gamma^2+\frac{4}{105}\right)\right].$$

Equation (25) is a third order expression in K and thus it is possible to obtain both a minimum and a maximum in the distribution. For $|\alpha_{21}/35\alpha_{01}|\approx 1.7$ we, in fact, obtain a maximum near $K_m/2$ and a minimum near $0.8K_m$, corresponding to the data. For $|\alpha_{21}/35\alpha_{01}|<1.4$ both maximum and minimum vanish leading to poor agreement with the first and the second to last exper-

imental points. For $|\alpha_{21}/35\alpha_{01}| > 2.2$ only the maximum remains and $F(K_m/2)/F(K_m)$ is too large. Thus, the useful values of $|\alpha_{21}/\alpha_{01}|$ are closely bounded. In Fig. 5 we reproduce the energy distribution for $|\alpha_{21}/35\alpha_{01}| = 1.73$ and $\cos\varphi = 0.45$. The χ^2 test yields 0.1 percent.

The significance of this fit is very small, although not negligible. However, it should be noted that the low significance is entirely due to the first point. Due to the small number of events expected in this region, the observed number is 3.8 half-widths out. The difference between the actual and expected number is only 15. Thus, if a few events are spurious (for instance, if a pion radiates) or if there is an unknown bias in favour of detecting slow negative pions, the fit is greatly affected.

From equation (24) it can be seen that the coefficients of $\cos^2\theta$ and $\cos^4\theta$ vanish for almost the same value of $\cos\varphi = 0.26|\alpha_{21}/35\alpha_{01}|$. Thus sufficient isotropy can be maintained within the useful range of $|\alpha_{21}/\alpha_{01}|$. For $|\alpha_{21}/35\alpha_{01}| = 1.73$, $\cos\varphi$ must be within ± 0.05 of 0.45. We plot the angular distribution for these values in Fig. 2. The χ^2 test yields 99 percent.

6. – More Detailed Analysis of the Data.

We have previously examined only the projection of the two-dimensional $F(K,\theta)$ distribution on the two axes. The amount of data which has been collected makes it statistically feasible to examine smaller regions of the (K,θ) area. In particular it is possible to split the region into two intervals of $K(\theta)$ and then examine the $\theta(K)$ distribution integrated over the values of $K(\theta)$ in each interval.

As the results of Sect. **3-5** are inconclusive, a more detailed examination of the above type is made to clarify the relative significance of the [0 −], [1 +] and [3 +] cases. The relevant formulae are only attained classically, as the statistics are not yet sufficiently good to warrant the inclusion of relativistic corrections in this section.

The most sensitive test has been obtained by considering the two energy intervals $K < K_m/2$ and $K > K_m/2$. The experimental data, as shown in Fig. 6, indicate significant anisotropy for the split interval; as opposed to the isotropy of the totally integrated case. For $\cos\theta \approx 0.3$ there is a ratio of 2 between the $K > K_m/2$ curve and the $K < K_m/2$ curve. The Göttingen [3] data are not included as only the totally integrated data were available to us.

The [0 −−] case (for any α_{22}/α_{00}) does not have any change in form between the totally and partially integrated $\cos\theta$ distribution. This is because of the symmetry of its K distribution about $K_m/2$. For $\alpha_{22} = 0$ the χ^2 test yields 10 percent significance for the $K < K_m/2$ angular distribution.

The [1 +] case which is classically isotropic for the totally integrated distribution ($\cos\varphi = 0.216\,|\alpha_{21}/35\alpha_{01}|$) reproduces the experimental features in this respect,

$$(26)\qquad F_{1+}\left(\cos\theta;\,K<\frac{K_m}{2}\right)=0.575\left[1+0.28\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|^2\right]^{-1}\cdot\left[\left(1+0.566\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|^2\right)-0.25\left|\frac{\alpha_{21}}{35\,\alpha_{01}}\right|^2\cos^2\theta\right],$$

where the normalization is such that

$$F_{1+}\left(\cos\theta;\,K<\frac{K_m}{2}\right)+F_{1+}\left(\cos\theta;\,K>\frac{K_m}{2}\right)=2F_{1+}(\cos\theta)\,.$$

The curve for $|\alpha_{21}/35\alpha_{01}|=1.73$ is reproduced in Fig. 6.

There is an anisotropy with the right sign and with ~ 50 percent of the experimental magnitude, and with an extremum at the correct value of $\cos\theta$. In this case the χ^2 test yields 37 percent for the $K<K_m/2$ angular distribution.

The [3 +] case which is classically isotropic for the totally integrated distribution ($\cos\varphi = -0.26\,|\alpha_{21}/\alpha_{03}|$) reproduces the splitting, between the $K \gtrless K_m/2$, very well for $\cos\theta \approx 1$, but by too small a magnitude for $\cos\theta\approx 0.3$.

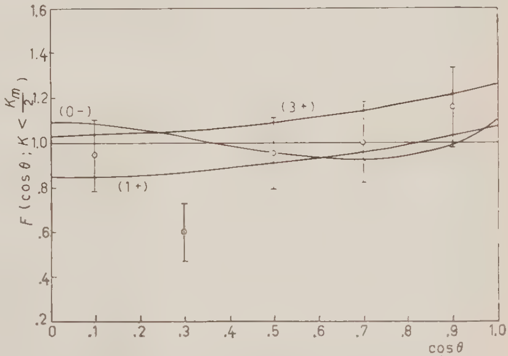

Fig. 6. – The classical angular distribution for $K>K_m/2$. The two terms of [0 −], [1 +] and [3 +] cases are plotted. The first term of [0 −] case is the horizontal line $F=1$. The coefficients are as in Fig. 2.

$$(27)\qquad F_{3+}\left(\cos\theta;\,K<\frac{K_m}{2}\right)=0.178\left[1+1.84\left|\frac{\alpha_{21}}{\alpha_{03}}\right|^2\right]^{-1}\cdot\left[\left(1+13.0\left|\frac{\alpha_{21}}{\alpha_{03}}\right|^2\right)+3.05\left|\frac{\alpha_{21}}{\alpha_{03}}\right|^2\cos^2\theta\right],$$

where the normalization is as in equation (26). The curve for $|\alpha_{21}/\alpha_{03}|=1.43$ is reproduced in Fig. 6. In this case the χ^2 yields 5 percent for the $K<K_m/2$ curve.

When the intervals are chosen in the following way

Interval A $\qquad 0.3K_m < K < 0.7K_m$

Interval B $\qquad K < 0.3K_m\,, \quad K > 0.7K_m$

the data, as shown in Fig. 7, indicate no significant trend away from isotropy.

For the [0 −−] case interval A emphasizes the (2, 2) distribution. Thus for large α_{22} anisotropy occurs. For the maximum value of $|\alpha_{22}/\alpha_{00}|$ allowed by the totally integrated distributions ($|\alpha_{22}/1200\,\alpha_{00}| = 1$, $\cos\varphi = -0.15$) we have

$$(28)\qquad F_{0-}(\cos\theta;\, 0.3K_m < K < 0.7K_m) \propto 0.238 - 0.192\cos^2\theta + 0.201\cos^4\theta\,,$$

which is plotted in Fig. 7. The χ^2 test yields 45 percent. Thus, the possible values of $|\alpha_{22}/\alpha_{00}|$ are not more limited by this test.

For both the [1+] and [3+] cases the condition for totally integrated isotropy is identical with the condition for isotropy of the intervals A and B. Therefore, these cases fit the data well. The χ^2 test yields 42 percent for either of them.

We have examined the relativistic energy distribution for $\cos\theta < \frac{1}{2}$. As the results do not help in the discrimination between the three possible cases, we do not include them here.

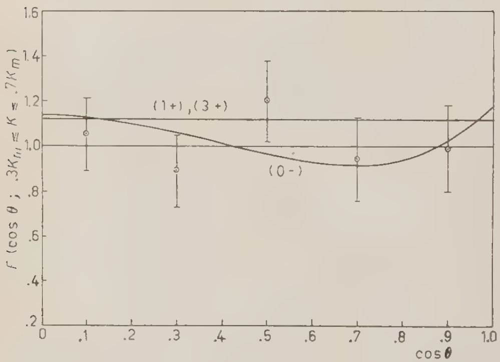

Fig. 7. – The classical angular distribution for $0.3K_m < K < 0.7K_m$, for the two terms [0 —], [1+] and [3+] cases. The [0 —] first term is the horizontal line $F=1$. The coefficients are as in Fig. 2.

7. – Discussion.

The combined results of Sects. **3-6** may be summarized by finding the total significance for the three independent distributions $F(K)$, $F(\cos\theta;\, K < K_m/2)$ and $F(\cos\theta)$. For the only cases which are plausible we have from the χ^2 test:

34 percent significance for [0 —], $\alpha_{22} = 0$;

1.5 percent for [1+], $|\alpha_{21}/35\alpha_{01}| = 1.73$ and $\cos\varphi = 0.45$;

4 percent significance for [3+], $|\alpha_{21}/\alpha_{03}| = 1.43$ and $\cos\varphi \doteq -0.5$.

Thus, on the basis of the present statistics a final decision is impossible. Several time the present number of events are required before it will be evident whether the low position of the first point in the $F(K)$ distribution or the large anisotropy in the $F(\cos\theta;\ K < K_m/2)$ distribution are statistical fluctuations or real effects. Due to the great importance of the first point of the energy distribution in determining the above significances, special attention should be paid to low energy negative pion events to determine if bias or secondary radiation or scattering play any role. If this one point is omitted from the χ^2 analysis one obtaines 44 percent for the [0], 41 percent for the [1+] and 20 percent for the [3+] cases given above.

It has been suggested that polarization of the decay plane with reference to the plane of τ-meson production can be used to distinguish between [0] and higher spins. While an experimental polarization would exclude [0], a negative result would yield no information as non-zero spins do not necessarily imply a large spin coupling (*). To discriminate between [1+] and [3+] would require the experimentally more difficult search for higher spin moments than that related to polarization (8). Again, a negative result would yield no information.

In I a lifetime argument is suggested to discriminate between various spins. However, the « natural lifetime » discussed (the time taken for the pions to pass through the angular momentum barrier and escape beyond the interaction distance) is in reality only a lower bound on the lifetime (+). The time of dissociation of the τ-mesons into pions must also be considered. This can be obtained only from the fundamental transition probability of the as yet unknown theory. Thus, while it is not true, as suggested in I, that the [0] lifetime should be shorter than the experimental lifetime, there is a strong argument against high spins which should have a « natural lifetime » larger than the experimental lifetime. From I it can be seen that for $\lambda \gtrsim 4$ the « natural lifetime » alone is equal to or greater than the experimental lifetime. Thus, τ-mesons of spin greater than 4 are impossible by this argument. Spin 4 is unlikely as this requires a very small dissociation time, On theoretical grounds the dissociation time is likely to be long as τ-meson decay falls into the class of « weak » interactions (9).

The above discussion together with the preceeding analysis leads to the following conclusions:

(*) This remark is also made in a preprint just received here (6).

(8) A. SIMON and T. A. WELTON: *Phys. Rev.*, **90**, 1036 (1953).

(+) This is also pointed out in reference (6). The numerical results obtained in this reference differ from those we obtain using I.

(9) M. GELL-MANN: *Proceedings of the Pisa Conference* (1955); R. G. SACHS: *Phys. Rev.*, **99**, 1573 (1955).

1) Relativistic effects are important and must be included in the analysis.

2) The angular momentum expansion of I is the most convenient means of carrying out the analysis, even if pion-pion interactions or other complicating effects are present.

3) If the interaction is assumed to be absolutely independent of l and l', then only the [0 —] assumption agrees sufficiently well with experiment.

4) If a weak l, l' dependence is admissible the [3+] assumption is almost as likely as the [0 —].

5) If a moderately strong l, l' dependence is allowed the [1+] assumption is also plausible.

6) The distinction between [0 —], [1+] and [3+] can most easily be made by examining new events with regard to:

a) The number of events for small K;

b) The partially integrated ($K < K_m/2$) angular distribution.

* * *

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We are also thankful to Miss P. ABRAHAMSON who helped in the calculations.

RIASSUNTO (*)

Si mette in evidenza la generalità del doppio sviluppo in serie del momento angolare per lo stato finale nel decadimento del mesone τ. Si fa rilevare il significato delle correzioni relativistiche. Si fa l'analisi relativistica dell'energia e della distribuzione angolare di 481 eventi. Si considerano uno o più termini dello sviluppo. Si esegue, inoltre, l'analisi classica di distribuzioni parzialmente integrate. In base a tale analisi si conclude che il mesone τ può avere [spin, parità] [0 —], [1+] o [3+] sulla base di un debole sistema di ipotesi riguardanti l'interazione. Se si utilizzano ipotesi più forti si può eliminare il caso [1+] o perfino i due casi [1+] e [3+].

(*) *Traduzione a cura della Redazione.*

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Decay Modes and Mean Life of Scattered K⁺-Mesons.

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Summary. — The investigation described aimed at finding any possible modifications in the relative frequency ratios of the different kinds of decay modes of positive K-mesons as a consequence of nuclear interactions. It has further been investigated whether a considerable number of the heavy mesons originating from nuclear interactions of K-mesons have a mean life substantially shorter than that observed in attenuation experiments. The negative results of our experiment together with other experimental data recently obtained seem to create difficulties for some of the theories on the properties of the τ- and θ-particles.

1. – Introduction.

It has been suggested that θ- and τ-decay correspond to intrinsically different particles. If these two particles had considerably different cross-sections for nuclear interactions such a difference ought to be detectable provided that both particles occur in the K^+-beam of the accelerator is sufficient quantity. If, on the other hand, one of them is predominant, and the other in the very short-lived decay-product of the first, as proposed by Lee and Orear [1], one might expect a reduction of the total mean life-time of heavy mesons having suffered a nuclear scattering, provided that the short-lived component can also be directly produced in nuclear reactions.

(*) On leave from Padua University.

(+) On leave from the Argentine Comision Nacional de la Energia Atomica.

[1] T. D. Lee and J. Orear: *Phys. Rev.*, **100**, 932 (1955).

In an experiment recently carried out by RITSON *et al.* [2] the relative frequencies of decay modes were determined for K^+-mesons which had suffered a nuclear scattering in the same target in which they had been created. The frequencies were not found to be varied with respect to the known frequencies of non-scattered K^+-mesons [3]. In that experiment, however, the comparatively long time spent by the mesons before they enter the emulsions prevents the detection of effects on the mean life-time.

In our experiment in which K^+-mesons scattered and arrested in the same stack are considered, evidence has been found neither for a modification in the ratios of decay modes nor for the seconadry production of an important fraction of short-lived K's.

A brief account of the experiment will be given as well as a discussion of some aspects of the results.

2. – Experiment.

Two stacks of Ilford G5 emulsions exposed to the K^+-beam of the Bevatron were scanned for interactions from which the meson is reemitted and coming to rest in the stack. 146 events were selected which had a primary K of between 50 and 110 MeV energy and a scattering angle greater than 40°. In about 70% of these collisions other prongs or a significant energy loss are also visible.

A particle undergoing a τ-decay (*) is emitted from 10 of the collisions. From the remaining scattered K's with one charged secondary two groups i) and ii) have been separated according to whether their secondary had an angle of dip

i) smaller than 12° or

ii) between 12° and 40°.

For the 31 tracks of group i) the grain-density has been accurately determined by counting in each case about 1000 grains. Careful calibrations with

(2) See D. M. RITSON's report at the VI Rochester Conference (1956).

(3) « G-STACK » COLLABORATION EXPERIMENT: *Nuovo Cimento*, **2**, 1063 (1955); D. M. RITSON, A. PEVSNER, S. C. FUNG, M. WIDGOFF, G. T. ZORN, S. GOLDHABER and G. GOLDHABER: *Phys. Rev.*, **101**, 1085 (1956); J. CRUSSARD, V. FOUCHÉ, J. HENNESSY, G. KAYAS, L. LEPRINCE-RINGUET, D. MORELLET and F. RENARD: *Nuovo Cimento*, **3**, 731 (1956); M. N. WHITEHEAD, D. H. STORK, J. R. PETERSON, D. H. PERKINS and R. W. BIRGE: UCRL 3295; G. ALEXANDER, R. H. W. JOHNSTON and C. O'CEALLAIGH: private communication.

(*) We denote as « τ-decay » the typical decay into three charged pions and as « τ-particle » or « τ-meson » the assumed pseudoscalar K-meson independent of its decay mode.

regard to minimum grain-density have been made for each event by using the pion tracks of the beam ($p \approx 350$ MeV/c; $I/I_{\text{plateau}} = 0.94$). Fig. 1 shows the results of the ionization measurements for the tracks of this group.

The 10 tracks with ionization greater than, or equal to, 1.05 times the plateau value have been submitted to scattering measurements and, if possible, followed to their ends. Two of them have been found to be μ-mesons from $K_{\mu3}$ decay while all the other 8 appear consistent with being due to secondaries of a $K_{2\pi}$ decay (+).

On 9 out of the 21 tracks of lower ionization the scattering has been measured. One was shown to be an electron; the results for the others are consistent with their being due to $K_{\mu2}$-secondaries.

Fig. 1. Distribution of the normalized grain-densities of 30 K-meson secondaries. One additional grey $K_{\mu3}$ secondary is not included in the histogram.

Out of the 55 tracks of group ii) 29 have been chosen which traverse at least $\frac{1}{2}$ of thickness of the emulsion layer in which the decay occurred. A grain-count was made on these such as to ensure a practically complete detection of tracks with an ionization $I/I_{\text{plateau}} \geqslant 1.3$ (minimum value for τ'-secondaries). Tracks with an ionization greater than, or near to, this lower limit have been followed. In this way another two examples of $K_{\mu3}$-decay have been found.

For obtaining the relative decay frequencies (*) it has been assumed that the scanning efficiency for all types of K's was 100% (the loss of fast K-se-

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(+) We suggest to write $K_{2\pi}$ and $K_{3\pi}$ for events in which all decay products are pions, and to reserve the notations $K_{\pi2}$ and $K_{\pi3}$ for cases in which the charged secondary is a pion while the neutral ones remain unspecified.

(*) The described procedure for identifying the different decay modes is very similar to that used by HOANG, KAPLON and YEKUTIELI [*Phys. Rev.*, **102**, 1185 (1956)].

condaries was found to be unimportant), that in group i) all K_{β} and $K_{\mu 3}$ have been identified, but in group ii) only 50% of the $K_{\mu 3}$. The results are shown in Table I, together with the average frequencies derived from various experiments in which non-scattered K^{+}-mesons were identified which had been produced under experimental conditions rather similar to those in our case.

It can be seen that the relative frequencies of the major decay modes $K_{3\pi}$, $K_{2\pi}$ and $K_{\mu 2}$ are not affected by the occurrence of nuclear collisions within the precision of our and the other experiments. The slight differences noticeable for the minor decay modes are statistically not significant.

In a total moderation time of $2.2 \cdot 10^{-8}$ s for all K-mesons showing a scattering angle greater than 40°, one single decay in flight has been found, a number not inconsistent with a unique mean life-time of $\sim 1.3 \cdot 10^{-8}$ s [4].

TABLE I.

Decay mode	Number of Events	Percentages	
		Present experiment	Non-scattered mesons (*)
τ	10	7	6.1 ± 0.3
τ'	0	0	1.8 ± 0.3
$K_{2\pi}$	8	23	25.5 ± 1.8
$K_{\mu 2}$	8 } (21)	55	58.6 ± 2.1
$K_{\beta 3}$	1 }	6	4.4 ± 1.0
$K_{\mu 3}$	4	9	3.6 ± 0.8

(*) These percentages are average values of the data of the « Richmann group » at Berkeley and of the Institute for Advanced Studies, Dublin, which were kindly communicated to us by the authors.

Stars possibly produced by K^{+}-mesons, from which no K-meson could be seen to emerge, have been carefully scrutinized for minimum ionization tracks. Such events showing the apparent emission of a relativistic particle instead of the scattered K-meson could result if a secondary K-meson was produced which decayed within a time shorter than $\sim 10^{-14}$ s (*). The few events of

(4) L. W. ALVAREZ, F. S. CRAWFORD, M. L. GOOD and M. L. STEVENSON: *Phys. Rev.*, **101**, 503 (1956); V. FITCH and R. MOTLEY: *Phys. Rev.*, **101**, 496 (1956).

(*) The existence of such events could also be interpreted as indicating the occasional violation of the rules of Gell-Mann, Nishijima and others, or the formation of fragments containing a K-meson, as suggested by PAIS and SERBER.

this type which had been found were demonstrated, however, to be background stars from which a grey proton track had accidentally been emitted in a direction parallel to that of the K-beam.

3. – Discussion.

Under the presuppositions mentioned in the Introduction the experimental results indicate that τ- and θ-mesons do not have appreciably different cross-sections for nuclear interactions. In addition there is also no indication for the reduction in the total mean life-time of scattered K-mesons which would follow from the theory of LEE and OREAR if one assumes that the short-lived, secondary K-mesons can be directly produced in nuclear reactions.

The failure to detect any low-energy γ-radiation associated with K-meson decay [5], and the very small mass difference [6] between τ- and θ-particles which would at present still be compatible with the experimental data from the Q-values make in addition theories of the type suggested by LEE and OREAR appear rather unsatisfactory.

YANG and LEE [7] have recently proposed a theory in which τ and θ form a parity doublet typical for all particles of odd strangeness. In this theory the two particles are expected to behave in a very similar way with respect to the strong interactions, but not necessarily with respect to the weak ones.

According to SALAM [8] the assumption that θ and τ are produced in equal numbers, as the theory of Yang and Lee would predict, and that the two mesons have equal decay constants for leptonic decay ($K_{\mu 2}$, $K_{\mu 3}$, K_{β}) would lead to a ratio of at least 1.4 for the two total decay constants. This value is not in good agreement with the present results on the mean lifetimes [9].

To assume that τ-particles are produced less frequently than θ-particles so that, with equal decay constants for leptonic decay and equal mean lifetimes, one obtains the experimental frequency ratio for $K_{3\pi}$ and $K_{2\pi}$ would be in disagreement with one of the prior assumptions of the theory of Yang and Lee, and difficult to reconcile with the results of our scattering experiment because of the implied differences of behaviour with respect to strong interactions. Another way to overcome the difficulty would be to assume that the leptonic decay constants are different for τ- and θ-particles. In this case,

(5) See L. W. ALVAREZ report at the VI Rochester Conference (1956).

(6) See report of the « Richman group » at the VI Rochester Conference (1956) and UCRL 3295.

(7) See (4) and also G. HARRIS, J. OREAR and S. TAYLOR: *Phys. Rev.*, **100**, 932 (1955).

(8) T. D. LEE and C. N. YANG: *Report of the VI Rochester Conference* (1956).

(9) A. SALAM: private communication.

however, it would have to be explained why the leptonic decays supplement the $K_{3\pi}$- and $K_{2\pi}$-decays so exactly that a single mean life-time results.

In conclusion it may be remarked that the existence of at least two different kinds of K-mesons which is suggested by the DALITZ analysis and the lack of polarization effects, furthermore the absence of any detectable differences between the K-mesons in their behaviour with regard to strong interactions and in their mean life-times, and the large difference between the frequencies for the decay modes $K_{2\pi}$ and $K_{3\pi}$ make the present theories on heavy meson decay appear not yet very satisfactory.

* * *

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M.C. thanks Prof. A. ROSTAGNI for granting him leave of absence.

The Deutsche Forschungsgemeinschaft has supported our work by the purchase of microscopes.

RIASSUNTO

In questo lavoro si cerca di mettere in luce una eventuale modificazione tra i rapporti dei diversi modi di decadimento dei mesoni K positivi in conseguenza di una interazione nucleare. Si cerca inoltre di stabilire se per effetto della interazione abbiano origine talvolta mesoni pesanti di vita media sostanzialmente più breve di quella osservata nelle esperienze di attenuazione. Il risultato negativo di questa ricerca, insieme ad altri dati oggi noti, permette di infirmare alcune delle teorie che cercano di spiegare il comportamento del doppietto di particelle τ e θ.

Sull'impiego dei radioisotopi per la determinazione dell'usura di utensili da taglio.

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(ricevuto il 15 Giugno 1956)

Riassunto. — Per la misura di usura di utensili da taglio col metodo dei traccianti radioattivi, si esamina l'opportunità di passare in soluzione il truciolo e di eseguire i conteggi in fase liquida, onde evitare correzioni per la geometria della sorgente, l'autoassorbimento ed il backscattering. La misura assoluta dell'usura si effettua paragonando a parità di ogni condizione l'attività di una soluzione di un frammento dell'utensile attivato con l'attività della soluzione del truciolo. Come esempio vengono riportati i risultati ottenuti su punte da trapano di diverse caratteristiche meccaniche fatte lavorare in identiche condizioni.

1. – Da alcuni anni il metodo dei traccianti radioattivi è stato applicato alla misura dell'usura di utensili da taglio. Da un'analisi della scarsa bibliografia reperibile risulta che, nelle sue linee generali, il metodo consiste nel rendere radioattivo per irraggiamento con neutroni lenti l'utensile e nel misurare, sul truciolo risultante dalla lavorazione, la radioattività che vi è restata aderente. Da questa misura si può risalire all'usura, poichè del materiale perso dall'utensile più del 90% resta aderente al truciolo [1].

I lavori che trattano l'argomento [1-5], e che si riferiscono unicamente alla usura di ferri da tornio, differiscono nelle modalità di esecuzione delle misure

(1) B. COLDING and L. G. ERWALL: *Nucleonics*, **11**, Febbraio 46 (1953).
(2) M. E. MERCHANT, H. ERNST and E. J. KRABACHER: *Trans. Am. Soc. Mech. Engrs.*, 549 (Maggio 1953).
(3) M. E. MERCHANT and E. J. KRABACHER: *Journ. Appl. Phys.*, **22**, 1507 (1951).
(4) M. E. MERCHANT: *Atomic Energy in Industry. Special Conf.* (1952). *Natl. Ind. Conf. Board, Inc.*
(5) E. J. TANGERMAN: *Am. Machinist*, **95**, 100 (1951).

di attività. COLDING *et al.* (1) utilizzano la radiazione β: preparano la sorgente prelevando pezzi di truciolo ed incollandoli su pezzi di perspex, in modo da ottenere una specie di mosaico di forma e dimensioni costanti, e la pongono a distanza fissa da un contatore a finestra. Questo procedimento è applicabile ogni qualvolta si abbia a che fare con trucioli riducibili ad una sorgente di geometria costante, per la quale non è necessario introdurre correzioni per il backscattering e per l'autoassorbimento. MERCHANT *et al.* (2) immergono un Geiger per γ direttamente in un bicchiere contenente i trucioli, rinunciando a rivelare la radiazione β.

Nel caso più generale i trucioli possono non presentare le caratteristiche richieste per l'applicazione del procedimento di Colding. Il campo di applicazione del procedimento di Merchant è limitato invece dal fatto che, in relazione agli isotopi formatisi nell'utensile, può risultare vantaggiosa la misura della radiazione β invece di quella γ.

Noi suggeriamo un procedimento che è indipendente dalla forma dei trucioli prodotti e dalla natura degli isotopi presenti nell'utensile e che permette, inoltre, una determinazione assoluta dell'usura.

Il nostro procedimento consiste nel passare in soluzione il truciolo. Per la misura dell'attività, quando sia conveniente contare la radiazione β, si usa un contatore a liquido con pareti di circa ~ 15 mg/cm^2, quando sia preferibile rivelare la radiazione γ si usa un fotomoltiplicatore con un cristallo opportunamente foggiato. Se si ha cura di mantenere uguali volume e densità delle soluzioni contate, non è necessario apportare correzioni per l'autoassorbimento e per il backscattering della radiazione β.

Inoltre, misurando contemporaneamente l'attività di una soluzione di uguale densità fatta con un frammento, di peso noto, dell'utensile, si determina l'attività specifica di questo (relativa alla geometria usata) e si risale alla misura assoluta dell'usura senza tener conto del decadimento dell'attività.

2. – A titolo sperimentale abbiamo applicato il procedimento alla misura dell'usura di punte da trapano. In questo caso, il metodo di Colding (1) non è manifestamente applicabile, dato che parte del truciolo è in forma polverulenta e la distribuzione delle particelle attive su di esso è irregolare.

L'acciaio delle punte utilizzate corrisponde a due tipi: *R* e *SR*, il primo contenente 9.1 % di W, e, in tenore decrescente, Cr, V e Mo; il secondo contenente il 18.1 % di W, e praticamente identico al primo per gli altri componenti. Le punte furono irraggiate per due settimane nel reattore di Harwell ($1.2 \cdot 10^{12}$ neutroni termici/cm^2 s). Si noti che la dose di neutroni veloci presa complessivamente dalle punte durante tale irraggiamento, anche supposto il flusso di neutroni veloci pari a quello dei neutroni lenti (mentre si può stimare in ogni caso minore di almeno un fattore dieci), è ancora inefficace a determinare alterazioni sensibili delle proprietà meccaniche.

I radioisotopi prodotti negli acciai *R* e *SR* da questo irraggiamento sono sostanzialmente: ^{185}W, ^{187}W, ^{59}Fe, ^{51}Cr con caratteristiche come dalla tabella I. Tenuto conto del tipo di decadimento, delle sezioni di attivazione, dei periodi di dimezzamento e del tempo intercorso fra la fine dell'irraggiamento e l'inizio delle misure, il radioisotopo che si presta meglio ad essere utilizzato risulta essere il ^{185}W per la sua radiazione β.

Perchè misure di usura di punte da trapano abbiano un senso definito, è evidentemente necessario precisare il più possibile le condizioni effettive di

TABELLA I.

Radioisotopo	$T_{\frac{1}{2}}$	Tipo di decadimento	Energia dei β (MeV)	Energia dei γ (MeV)
^{185}W	76 d	β^-	0.426 — 0.370	0.056
^{187}W	24.1 h	β^-	1.33 — 0.63	0.696 — 0.618 — 0.480 — 0.138 — 0.078
^{59}Fe	46.3 d	β^-	0.46 — 0.26	1.30 — 1.10
^{51}Cr	26.5 d	E. C.	—	0.33 (8 — 10%)

lavoro delle punte stesse e condurre un gran numero di prove. Noi ci limitiamo, in queste misure di prima approssimazione, a saggiare il procedimento su due sole punte che chiameremo R e SR. Circa le condizioni di lavoro: le punte lavoravano a carico costante, realizzato applicando il carico mediante un peso agente su una puleggia sostitutiva della leva ordinaria del trapano: i fori, ciechi, profondi 10 mm, erano praticati su una spessa lastra di acciaio, tanti quanti ogni punta era in grado di eseguire senza aumentare il carico; diametro delle punte: 2 mm; trapano sensitivo Olivetti.

I trucioli provenienti da ciascun foro venivano raccolti separatamente, accuratamente pesati e sciolti, a caldo, in acqua regia. Tutte le soluzioni erano portate a uguale concentrazione. Si misurava quindi l'attività di 8 cm³ di ogni soluzione per un intervallo di tempo di 3 minuti mediante un contatore per liquidi. Ai risultati venivano applicate le usuali correzioni per il fondo e per il tempo morto del contatore.

In tal modo si sono ottenute misure dell'attività relativa persa dalle punte durante le singole forature. Per passare dai valori relativi a quelli assoluti, e risalire quindi all'usura effettiva, da esprimersi in g di sostanza perduta per foro, si è misurata l'attività specifica delle punte nel modo indicato in precedenza. Le misure di attività specifica sono state fatte contemporaneamente alle misure di usura, nel senso che l'intervallo di tempo intercorso fra l'esecuzione delle due serie di conteggi è assai breve rispetto al periodo di dimezzamento dell'attività delle punte.

Usando la precauzione di mantenere uguali le densità delle soluzioni e di eseguire i conteggi contemporaneamente, non è necessario apportare correzioni nè per l'autoassorbimento, nè per il backscattering, nè per il periodo di dimezzamento, come è già stato detto. Le misure sono state eseguite a partire da dieci giorni dal termine del periodo di irraggiamento delle punte, in maniera che l'attività del ^{187}W fosse quasi completamente decaduta.

Le misure fatte sono riportate per le due punte R e SR nella Fig. 1 e le

TABELLA II. - *Usura media/foro in 10^{-6} g, determinata sul tratto a minor pendenza della curva di usura totale.*

punta	att. spec. imp./mg. min	usura media/foro
R	6 950	5,6
SR	11 539	1.7

usure medie per foro subite sono riportate nella Tabella II. insieme alle attività specifiche.

3. – Benchè queste misure sieno intese più a dare un esempio del metodo proposto, che a fornire informazioni di carattere tecnologico, si possono fare alcune ovvie considerazioni sulla Fig. 1, che mostra l'andamento dell'usura totale in funzione del numero dei fori eseguiti. Da essa si può vedere come

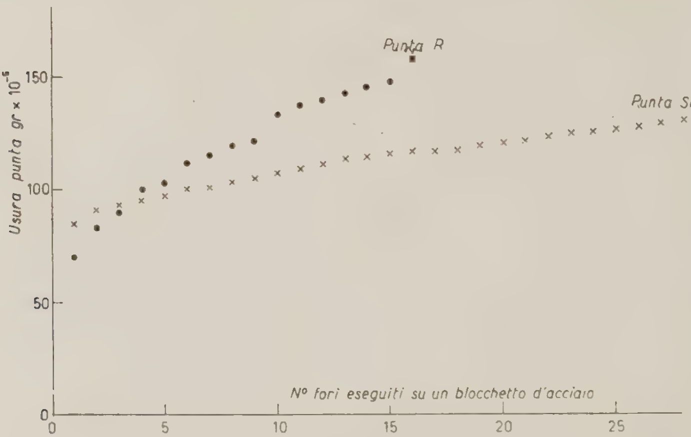

Fig. 1.

l'usura specifica (usura/foro) sia massima all'inizio e poi diminuisca rapidamente e tenda ad un valore costante; esiste, cioè, dopo i primi fori, una relazione pressocchè lineare fra l'usura totale della punta e il materiale totale estratto. Si notano, talora, bruschi scarti dalla linearità: tali deviazioni si devono attribuire ad improvvise rotture del filo della punta, come può anche verificarsi direttamente osservando il filo con un microscopio. La pendenza del primo tratto dei grafici è, nell'uno e nell'altro caso, indice della perdita del filo morto della punta. Di conseguenza, questo primo tratto non può essere considerato significativo, se non nel quadro di un'indagine approfondita: l'usura vera e propria viene valutata sul secondo tratto.

Assunta la pendenza di questa parte della curva come misura della velocità di usura della punta, si può osservare che l'acciaio super-rapido (SR) presenta, rispetto all'acciaio rapido (R), un'usura inferiore di un fattore circa 0.3, in accordo col fatto che il primo ha una percentuale di W (18.1 %) molto superiore a quella del secondo (9.1 %).

Dai valori delle attività specifiche riportate in Tabella II, considerando già significativi i conteggi superiori al doppio del fondo, che si aggirava sui 10 ÷ 15 imp/min, si vede che la sensibilità del metodo è compresa, nel nostro caso, fra 1.5 e 3.5 per 10^{-6} g. Questa sensibilità può essere agevolmente migliorata di un fattore 2 o anche 3, aumentando il flusso neutronico e la durata dell'irraggiamento.

Il metodo di conteggio in fase liquida consente una buona riproducibilità delle condizioni sperimentali, non presenta difficoltà ed è rapido, qualora le operazioni ad esso connesse vengano eseguite in serie. L'apparecchiatura di conteggio è la più semplice e convenzionale possibile e relativamente poco costosa.

Infine, i pericoli derivanti dalla radioattività, data la bassissima attività dei trucioli, si riducono a quelli dovuti all'irraggiamento esterno da parte delle punte durante l'operazione di foratura: tale pericolo può essere agevolmente annullato, tenendo gli operatori a distanza opportuna e schermando le punte col disporre sul piano del trapano un piccolo muro di Pb di 5 cm di spessore.

Adottando queste precauzioni, la dose ricevuta dai nostri operatori nel corso del lavoro è stata inferiore ad 1/10 della dose/ora permessa, stabilita in 6 mr/ora. In base a queste considerazioni pensiamo, quindi, che il metodo abbia buone possibilità di applicazione pratica nel campo industriale.

* * *

Ci è gradito ringraziare il prof. G. Bolla per l'interesse mostrato al presente lavoro e le utili discussioni avute in proposito.

SUMMARY (*)

For the measure of the wear of the cutting tools by the method of radioactive tracers the expediency is examined to solve the cuttings and to do the counting in the liquid phase in order to avoid corrections due to the geometry of the source, to self absorption and backscattering. The absolute measure of the wear is done by comparing, all conditions being equal, the activity of the solution of the fragment of the activated tool with the activity of the solution of a chip. To exemplify, results are reported obtained on drills of different mechanical characteristics having worked in identical conditions.

(*) *Editor's Translation.*

Semi-Automatic Recorder for Filar Micrometer Eyepiece and Its Application to Track Measurement.

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(ricevuto il 25 Giugno 1956)

Summary. - A differential transformer has been mounted on a filar micrometer eyepiece and has been coupled to a recording circuit, to make possible automatic recording of the hairline positions. The apparatus greatly reduces the time required to make measurements with a filar eyepiece; e.g., as when measuring the multiple Coulomb scattering of tracks in nuclear emulsions.

The filar micrometer eyepiece has been widely used for making measurements on tracks in nuclear emulsions, and data are usually taken with it in the following manner. The observer moves a hairline to a particular point on a track. He then reads the position of the hairline from a scale in the field and a micrometer drum. Finally, he must record the reading. Thus, a considerable fraction of the measurement time is spent in reading and recording the data as well as in readjusting one's eyes to the microscope. We have designed an apparatus which records the positions of the hairline electromechanically. Using this instrument, the observer need only position the movable hairline; then, by closing a switch, he automatically records the position of the hairline as a number on a paper tape (1).

We have mounted a differential transformer (2) onto the body of the eyepiece, as shown in Figs. 1 and 2. Its movable armature, *A*, Fig. 2, is rigidly

(1) A. G. Ekspong has reported an apparatus which graphically records first differences between successive positions of the hairline in a filar eyepiece (*Ark. f. Fys.*, **9**, 49 (1954)).

(2) ATCOTRAN differential transformer, Class 6208; Automatic Temperature Control Co., Philadelphia, Pennsylvania.

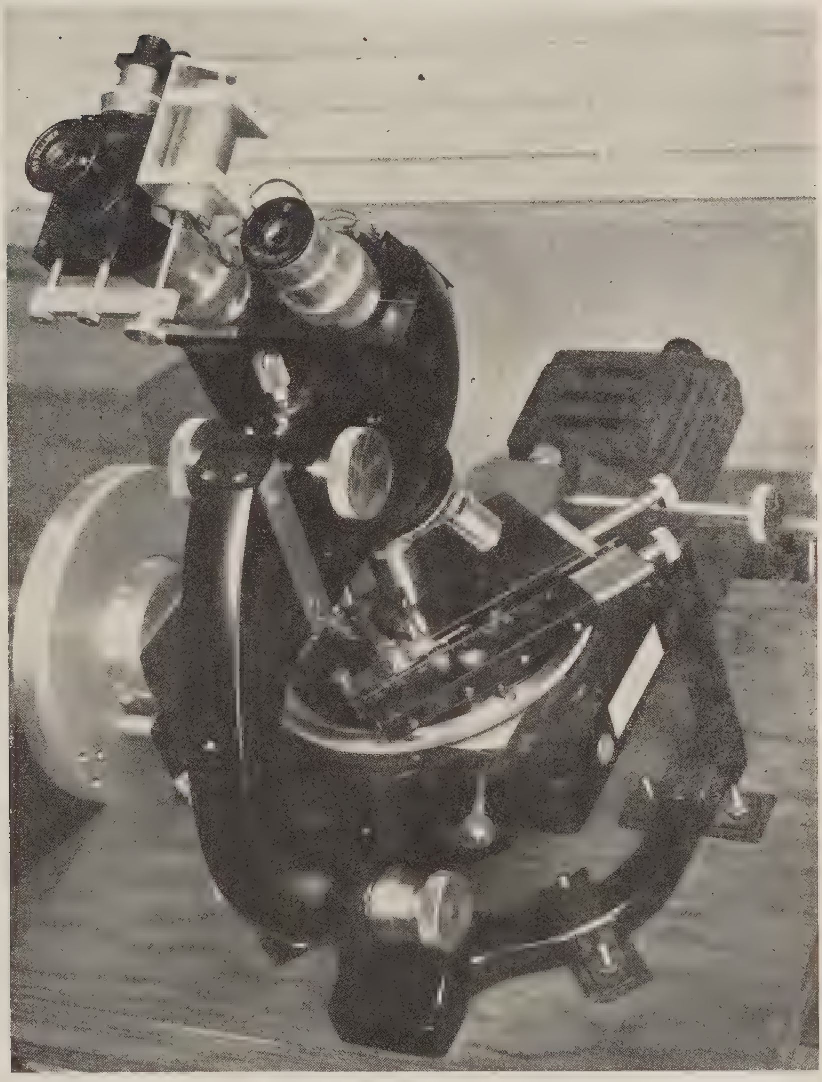

Fig. 1. Filar eyepiece and differential transformer mounted on Bausch and Lomb microscope.

connected by B to the sliding carriage which carries the hairline. This produces a one-to-one linear relationship between the positions of the hairline and the armature. Calibration of the filar is therefore independent of irregularities in the pitch of the screw, which moves the hairline and of variations in the spacing of filar divisions. It depends, instead, on the linearity of the automatic recording system.

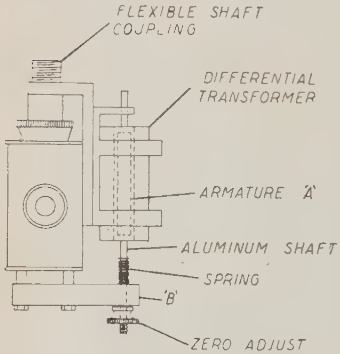

Fig. 2. – Schematic drawing of eyepiece mounting for differential transformer.

Fig. 3 is a block diagram of this system. P_1 and P_2 are the primary coils of T_P, the pickup transformer, while P_3 and P_4 are the primary coils for T_B, the balancing transformer which is used to null signals from T_P. They are energized by a 6 V 60 Hz source. The two secondary coils of each transformer produce output voltages which are 180° out of phase with each other. When the armature is in the middle of transformer T_P, the flux distribution between the paired coils P_1-S_1 and P_2-S_2 is the same, and the output voltages from S_1 and S_2 are equal but of opposite phase. Therefore, no signal is produced by the pickup transformer; the same condition exists in the balancing transformer. When the observer rotates the drum, moving the armature of T_P along its axis, the flux distribution changes in S_1 and S_2 and one produces a larger voltage than the other. Therefore, a voltage, V_1 appears from T_P. However, the output of T_P is also connected to the output of T_B but again in opposite phase. Since the armature of T_B is still in the middle of its windings, no voltage appears from T_B and the full signal from T_P enters the amplifier. The amplified voltage is used to turn a servo motor through an angle which depends on the applied voltage and in a direction which depends on its phase. The servo motor rotates a cam, raising a lever which moves the armature of T_B in the same direction as the armature of T_P. A voltage, V_2, now appears at the output of T_B but it is opposite in phase to V_1 and when $V_2 = V_1$, the signal to the amplifier is zero and the servo motor stops. It has also positioned a digital converter, which can be set in 999 different commutator positions, each corresponding to a three–digit number between 1 and 999. The observer

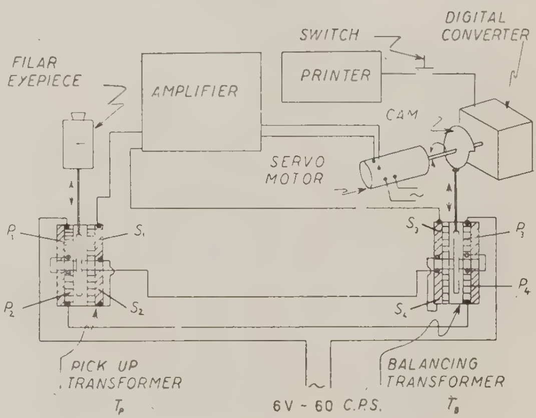

Fig. 3. Block diagram of recording circuit.

closes a switch which de-energizes the servo motor and energizes the solenoid-operated printer, thus recording a number corresponding to the position of the hairline.

To investigate the behavior of the recording system, we selected a 7 cm track produced by a relativistic α-particle in a 400 μm G.5 plate. Multiple Coulomb scattering measurements made with a Koristka MS-2 microscope give it an apparent mean scattering angle $\bar{\alpha} = 0.002°/100\ \mu\text{m}$, at 5 000 μm cells. Therefore, reading at 50 μm cells would measure only spurious scattering, or « noise ». The converted filar eyepiece was mounted on a special Bausch and Lomb scattering microscope and 100 measurements were made on the α-particle track, at 50 μm intervals. Visual readings were taken from the micrometer drum, to measure the noise contributed collectively by the emulsion, microscope and observer. Simultaneous automatic readings were taken to measure the same noise plus the noise of the recorder. Three sets of measurements were made for three sensitivities of the servo-mechanism. The results are shown in Table I. $\bar{\bar{D}}$ is the mean 2nd difference obtained by

TABLE I. - *Comparison between automatic and visual readings of scattering measurements on the track of a relativistic α-particle.*

	$\bar{\bar{D}}$ (μm) 1	$\bar{\bar{D}}$ (μm) 2	$\bar{\bar{D}}$ (μm) 3
Automatic Readings	0.092 ± .009 (1 div = .095 μm)	0.063 ± .006 (1 div = .048 μm)	0.067 ± .007, (1 div = .019 μm)
Visual Readings	0.066 ± .007 (1 div = .095 μm)	0.062 ± .006 (1 div = .095 μm)	0.061 ± .006 (1 div = .095 μm)

the Fowler co-ordinate method ([3]). To understand these results, it should be recalled that a digital converter has an inherent mechanical uncertainty of one division, whereas visually, the observer can read the micrometer to fractions of a division. Thus the higher noise, column 1, Table I, produced by the automatic recorder when running at its lowest sensitivity, (1 div. = 0.095 μm) is partly due to this random source of error. At higher sensitivities, the absolute value of the error introduced by the converter approaches the error of the observer in reading fractions of a drum division, and the noise values are equivalent, as shown in columns 2 and 3 of Table I. However, the resolution of the automatic system is limited, in the present instrument, by the sensitivity of the pickup transformer and the linearity of the cam in the servo-mechanism. We are exploring other pickups and circuits in an effort to improve the resolution. We are also planning to feed signals from the digital converter to a tele-tape puncher so that the data may then be fed directly to a digital computer for differencing.

([3]) P. H. FOWLER: *Phil. Mag.*, **41**, 169 (1950).

The apparatus is currently being used in an analysis of cosmic ray α-particle tracks found in a pellicle stack exposed at the Galapagos Islands. Multiple Coulomb scattering measurements were made on a group of $\sim 4\times$ minimum ionization tracks, using a basic cell of 250 μm. The Bausch and Lomb microscope mentioned above was used, and simultaneous visual and automatic readings (1 div. = 0.048 μm) were taken. Table II summarizes the results for

Table II. *Scattering measurements on 22 tracks having $\sim 4\times$ minimum ionization.*

Track	N Number of Cells	Visual Reading $\bar{D}$ (μm)	Automatic Reading $\bar{D}$ (μm)	% Difference $100(b+c)/2$	% Error $100/\sqrt{N}$
	a	b	c	d	e
E-3809	19	1.238	1.247	+ 1	23
E-3804	21	0.150	0.120	− 11	22
S-3700	16	0.786	0.735	− 3	25
E-3706	29	0.131	0.134	+ 4	19
S-3612	27	0.752	0.734	− 1	19
E-3309	39	0.195	0.206	+ 2	16
S 3310	41	0.101	0.117	+ 7	16
S-2106	45	1.048	1.048	0	15
E-2102	19	0.725	0.718	− 1	23
S-2112	22	2.097	2.154	+ 1	21
E-507	14	0.652	0.646	1	27
S-2	43	0.111	0.107	1	15
E-3	36	0.204	0.208	+ 1	17
E-6	16	1.006	1.007	0	25
E-3215	27	0.343	0.335	− 1	19
E 3006	21	0.570	0.577	+ 1	22
E 3005	25	0.375	0.389	+ 2	20
E 2905	24	0.278	0.264	3	20
S-1902	36	0.163	0.168	+ 1	17
S-1601	29	0.377	0.380	+ 1	19
E-1605	30	0.280	0.288	+ 1	18
E 1001	45	0.143	0.146	+ 1	15

22 tracks. Column a gives the number of independent cells for each track. Columns b and c give the noise corrected $\bar{D}$ for each method of recording. In column d, the percentage deviation of the «automatic» value from the average of the two values is given. Column e gives the standard error for either $\bar{D}$, based on the number of readings. It is evident from column d that there is no systematic increase in $\bar{D}$ when the measurements are recorded automatically. In fact, a comparison of the last two columns shows that the small error introduced in some cases by the recorder, contributes negligibly to the overall error.

* * *

We should like to thank Dr. M. M. SHAPIRO for his very helpful advice and encouragement. Miss KATHRYN DEANGELIS was of great assistance in the taking and analysis of data.

RIASSUNTO (*)

Un trasformatore differenziale è stato montato su un oculare con micrometro a reticolo ed accoppiato a un circuito registratore, onde rendere possibile la registrazione automatica delle posizioni della linea di fede. L'apparecchio riduce notevolmente il tempo richiesto dalle misure con oculare a reticolo; ad esempio nella misura dello scattering coulombiano multiplo di tracce in emulsioni nucleari.

(*) *Traduzione a cura della Redazione.*

Ionization at the Origin of an Electron Pair of Very High Energy.

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(ricevuto il 22 Maggio 1956)

Recently two papers appeared concerning ionization caused by electron pairs of very high energy near their origins. A. E. ČUDAKOV (1) estimated theoretically the reduction of ionization caused by an electron pair (as compared with ionization produced by two particles separately) on the length on which the separation between negaton and positon is smaller than the maximum value of the interaction radius for ionization. Assuming for this value $5\cdot10^{-7}$ cm and neglecting the influence of the relative scattering of the two particles A. E. ČUDAKOV obtained a formula from which follows that for energies of pairs of some 10^{11} eV we can expect measurable effects of reduction of ionization on distances of some hundreds microns. Quite independently from ČUDAKOV, D. H. PERKINS (2) following a suggestion of D. T. KING (1950) (*) made ionization measurements on a number of electron pair tracks with energies of $\sim 10^{11}$ eV. It follows from his measurements that for pairs of this order of magnitude this effect is measurable for distances $\lesssim 100\,\mu$m from the origin of the pair. In consequence of poor statistics on such length it was only possible to measure this effect as a mean effect for several pairs. PERKINS has performed these measurements for 7 pairs.

The aim of this letter is to report measurements of ionization on a single pair of energy probably higher than the pairs of PERKINS. The pair under discussion is the beginning of a large electron-photon cascade which at the distance of ~ 2.5 cascade units contains ~ 70 tracks of electrons of energies higher than 10^8 eV, from what we get roughly for the energy of the first pair some 10^{11} eV (3). On a length of 9 540 μm the track is single. At this distance from the origin, an apparent trident of high energy is generated and the four particles form a single track for further se-

(1) A. E. ČUDAKOV: *Izw. Akad. Nauk USSR Ser. fiz.*, **19**, 651 (1955).

(2) D. H. PERKINS: *Phil. Mag.*, **46**, 1146 (1955).

(*) Unpublished.

(3) A. JURAK, M. MIĘSOWICZ, O. STANISZ and W. WOLTER: *Bull. de l'Acad. Pol.*, Cl. III, 369 (1955); M. MIĘSOWICZ, W. WOLTER and O. STANISZ: communicated at the Pisa Conference, June 1955.

Added in proof; The recent analisis of the electrons energy spectrum at the depth 2.5 c. u give for the energy of primary photons the value $(70^{+3.4}_{-2.6})\cdot10^{11}$ eV.

veral hundreds microns. We could only estimate the upper limit of the value of the projected angle of this pair as $1.7 \cdot 10^{-5}$ radians. Information about the energy of such pairs is very limited. We are able to measure only the projected $\sim 10^{11}$ eV, which seems to be very underestimated.

As the measure of ionization we have taken the coefficient g of the exponential distribution of gap lengths after FOWLER and PERKINS [6]. This distribution is

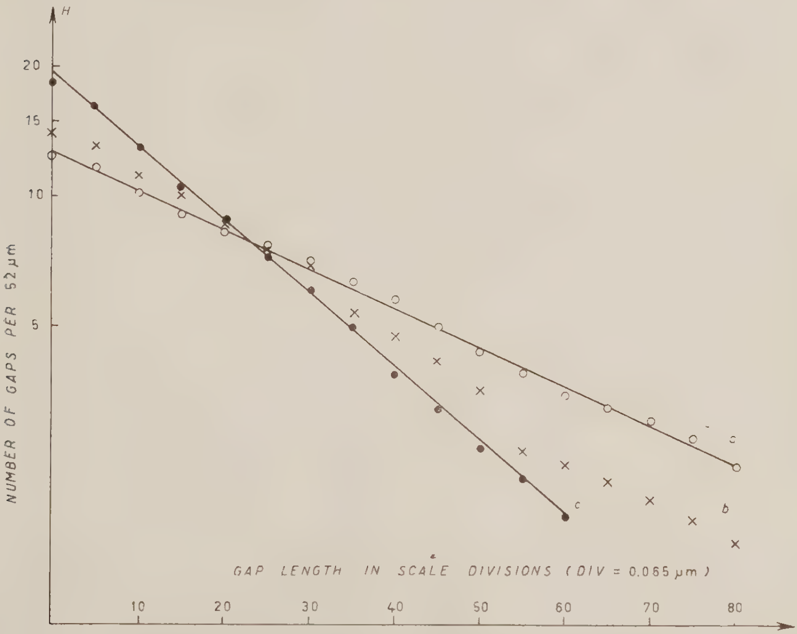

Fig. 1. – Gap length distributions on tracks of: a) a relativistic electron (plateau); b) the investigated pair ($0 \div 260$ μm); c) the investigated pair ($260 \div 1560$ μm).

angle of the pair. Besides, for distances of ~ 1 mm from the origin of the pair the separation of the electron tracks is essentially determined by multiple scattering (LOHRMANN [4]). In any case it is very probable that the opening angle of our pair is smaller than the value mentioned above. Assuming the equipartition of the energy we get from the formula of BORSELLINO [5] as a lower limit for the energy of the pair the value

given as $H = B \cdot e^{-gl}$ where H is the density of gaps of length greater than l and B is the blob density. The gap lengths were measured with a filar micrometer. There were carried out the following measurements of the distribution of gap lengths:

(a) For a relativistic electron (plateau) on the length of 1040 μm.

(b) for the investigated pair on the first segment of 260 μm from the origin of the pair.

(4) E. LOHRMANN: *Nuovo Cimento*, **2**, 1029 (1955).

(5) A. BORSELLINO: *Phys. Rev.*, **89**, 1023 (1953).

(6) P. H. FOWLER and D. H. PERKINS: *Phil. Mag.*, **46**, 587 (1955).

(c) For the investigated pair in the segment of track between 260 and 1 560 μm.

These distributions are shown on Fig. 1.

The investigated tracks were so long as compared with the « dip » that the gap length could be measured simply along the track. From these measurements the following values for the coefficient g of exponential gap length distribution can be obtained:

g-values per scale division ($=0.065$ μm)	
Relativistic electron (plateau)	0.021 ± 0.001
High energy pair near the origin ($0 \div 260$ μm)	0.028 ± 0.004
High energy pair far from the origin ($260 \div 1\,560$ μm)	0.039 ± 0.002

The values of errors are calculated as $1/\sqrt{N_B}$ where N_B is the number of blobs. In order to examine whether the g values were not influenced by local differences in development of the emulsion, control measurements of g were carried out for a long relativistic track nearly parallel to the track of the pair at a distance of ~ 50 μm on a length of ~ 1000 μm including the neighbourhood of the pair origin. On all segments of the track of 260 μm length, the same value of g was obtained within the limits of statistical errors. On Fig. 2.

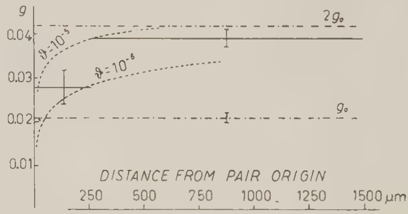

Fig. 2. – —— the values of g determined on two segments of track; –·–·– the values g_0 (plateau ionization) and $2g_0$; · · · · the curves of Čudakov for opening angle of the pair $\vartheta = 10^{-5}$ and $\vartheta = 10^{-6}$ radians.

are given the values of g for two segments of the pair. On the same figure the curves of ČUDAKOV for opening angles $\vartheta = 10^{-5}$ and $\vartheta = 10^{-6}$ radians are shown. If the observed effect of the diminishing of ionization near the origin of the pair is really caused by the unlike charges of negaton and positon then we can use this effect for the estimation of the opening angle of the pair which is very valuable because of difficulties in measurements of this angle in any other way. It follows from Fig. 2 that the value obtained in this way for the opening angle does not contradict with the value estimate from the energy development of the cascade and the Borsellino formula.

Unusual Cosmic Ray Events Observed in a Multiplate Cloud Chamber.

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Tata Institute of Fundamental Research - Bombay

(ricevuto il 4 Giugno 1956)

In an arrangement of multiplate cloud chambers triggered for penetrating showers at Ootacamund (7 500 ft. above sea level), two unusual events have been observed. In both the events, a particle comes to rest in one of the lead plates in the chamber and a charged secondary emerges from the plate with a range greater than 143 g/cm² of lead in one case, and a range greater than 185 g/cm² of lead in the other case. Among the known K-meson and hyperon decays or capture stars no definite case of an energetic secondary of range greater than 100 g/cm² of lead has so far been observed.

The experimental arrangement, as shown in Fig. 1, consists of two multiplate cloud chambers mounted one above the other. The top chamber is 17 cm deep and 40 cm in diameter and has seven half inch lead plates with stainless steel reflectors, each plate being equivalent in thickness to 15.6 g/cm² of lead (*). The bottom chamber is of size $60 \times 60 \times 25$ cm³ and contains thirteen lead plates of the same equivalent

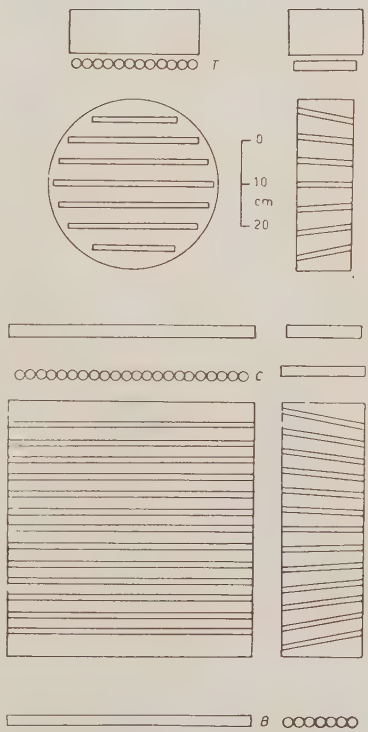

Fig. 1. Experimental arrangement.

(*) Uniformity of density was checked by weighing the plates.

thickness as above. In the figure, T, C, and B are trays of Geiger counters and the cloud chambers were triggered by coincidence selection systems T_4C_1 and

Fig. 2. – An S-particle emerging from an interaction in the first plate stops in the fifth plate. The fast secondary coming out of this plate goes towards the star in the left hand bottom corner.

C_4B_1. (Suffixes indicate the minimum number of counters discharged in each tray). About half of the total number of photographs were taken with both the chambers expanding simultaneously on receiving a pulse from either the coincidence T_4C_1 or C_4B_1. The rest of the photographs were taken with the two chambers expanding independently, the top by the system T_4C_1 and the bottom by C_4B_1. Forty eight thousand stereoscopic photographs taken so far with this set up have yielded a few (*) S-particles (1) of the usual type and these two particular events, the photographs of which are shown in Figs. 2 and 3, and are described in detail below.

In Fig. 2, the S-particle is produced in an interaction in the first lead plate of the bottom chamber and slows down as it penetrates the next three plates and appears to stop in the fifth plate, $9.7^{+1.3}_{-2.3}$ mm below the top of the plate. The visual ionization estimates, of greater than twice minimum for its tracks in the last two compartments before it stops, rules out the possibility of its mass being as low as that of a π-meson. However, from the ionization estimates in the four compartments in which the primary track is seen, one cannot rule out the possibility of its mass being as high as that of a hyperon. The time of flight of this S-particle is estimated to be about 10^{-9} s. From the fifth plate where this S-particle stops, there emerges a minimum ionizing secondary particle which penetrates seven lead plates and enters the last plate at minimum ionization. A star in the last plate at the left hand corner obscures the possible continuation of the track of the secondary in the compartment below the last

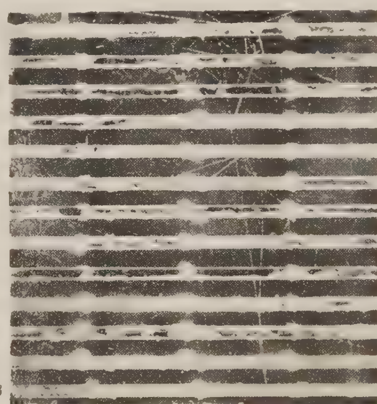

Fig. 3. – An S-particle stops in the thirteenth plate and gives rise to an upward moving secondary. The fast secondary penetrates at least eleven lead plates, produces an upward shower in the third lead plate and may be penetrating the top-most plate to escape the chamber. The bending of the louvres of the illumination system has made the tracks faint in the first four compartments.

(*) Eighteen cases of stopping K-mesons have been observed so far.

(1) M. Annis, H. Bridge, H. Courant, S. Olbert and B. Rossi: *Nuovo Cimento*, 9, 624 (1952).

lead plate. The secondary appears to produce the above mentioned star but it is actually 5 ± 1 mm away from the origin of the star. (5 ± 1 mm is the least projected distance between the origin of the star and the continuation of the particle in the lead as measured directly from the enlargements of the photographs.) Stereoscopic reprojection of the photographs of the event gives a range of 138 ± 7 g/cm^2 of lead for the penetration of the seven lead plates. To this we add a minimum of 4 g/cm^2 of lead for the estimated traversal of the secondary in the plate from which it emerges, and 8 g/cm^2 of lead because of the fact that the secondary is still less than twice the minimum ionization when it enters the last plate, thus giving a minimum total range for the secondary of 143 g/cm^2 of lead.

The second event, shown in Fig. 3, has an S-particle produced in a secondary interaction in the sixth plate. The S-particle slows down and stops in the last lead plate. The visual ionization estimates of greater than twice minimum for its tracks in the last two compartments before it stops, exclude its mass being that of a π-meson. Similar ionization estimates of less than twice minimum for its tracks in the first three compartments below the sixth plate, exclude this particle from having a mass as large as that of a Σ-hyperon; for if it were a particle of hyperonic mass, it would be ionizing twice minimum in all these three compartments. The time of flight of this particle also is estimated to be about 10^{-9} s. From the last lead plate where this S-particle stops, a minimum ionizing backward secondary is emitted, which penetrates at least eleven lead plates, and produces an upward shower in the third lead plate. The track of this secondary in the topmost compartment is not sufficiently clear to assert that the particle goes through the first lead plate and thus escapes the chamber. Stereoscopic reprojection of the photographs of the event gives a range of 175 ± 2 g/cm^2 of lead, for the penetration of eleven lead plates. To this, one has to add a minimum of 4 g/cm^2 of lead for the estimated traversal of the secondary in the thirteenth lead plate from which it emerges. (This value is obtained from the top limit of the ionization estimates of 3 to 5 times minimum for the track of the S-particle in the compartment above the last lead plate.) Finally, as before, we add 8 g/cm^2 of lead because the secondary is still less than twice minimum in ionization below the topmost lead plate. Thus we add up to a total minimum range of 185 g/cm^2 of lead for the secondary particle.

In both the events, the secondary minimum ionizing particles cannot be electrons since in each case, they have traversed several radiation lengths without producing any electronic shower. If they are both L-mesons, they are likely to be μ-mesons, since each has traversed about one nuclear interaction mean free path without producing any visible signs of interactions. The kinetic energies of these secondaries, corresponding to the above mentioned minimum ranges are 193 MeV and 239 MeV, assuming them to be μ-mesons.

Among the known K-meson decays and capture stars, no definite case of an energetic secondary of range greater than 100 g/cm^2 of lead has so far been observed. (However, a doubtful case of a secondary range > 100 g/cm^2 of lead has been reported by ARMENTEROS *et al.* [2]). The observed minimum values of 143 and 185 g/cm^2 of lead for the secondary ranges exclude the events as being similar to any of the so far observed K-meson decays or capture stars.

We consider the various possible

[2] R. ARMENTEROS, B. GREGORY, A. HENDEL, A. LAGARRIGUE, L. LEPRINCE-RINGUET, F. MULLER and C. PEYROU: *Nuovo Cimento*, **1**, 915 (1955).

interpretations for these events, namely, chance coincidence between a stopping proton and a fast particle disappearing in the plate, the decay in flight of a known K-meson, the nuclear capture of a known K^--meson, a new type of hyperon, the annihilation of an anti-proton [3], or a new type of a heavy K-meson.

The probability that an apparent S-event in our chamber is due to the chance coincidence between a stopping proton and a «stopping» minimum ionizing particle penetrating seven plates or more is given by

$$\frac{3.6\cdot 0.1}{12\cdot 1\,420}\approx 2\cdot 10^{-5}\,,$$

where 3.6 cm³ is the effective volume of the cell in the lead plate, anywhere in which due to errors of reprojection the two particles could meet, 12 is the number of plates in which an identified proton could stop, 1 420 cm³ is the volume of one lead plate and 0.1 is the number of «stopping» fast charged particles for each stopping proton observed. One thousand and sixteen stopping protons have been observed by us so far and so one expects 0.02 events due to chance coincidence, while actually we have observed two events.

On entering the last lead plate, if the S-particles have decayed in flight before stopping, then the events are compatible with the decays in flight of known K-mesons, ($K_{\mu 2}$, $K_{\pi 2}$, $K_{\mu 3}$). Assuming the lifetime of these K-mesons to be 10^{-8} s [4], the probability that an S-particle decays in flight in the last lead plate in which it could have stopped, and emits a secondary with a range $\geqslant 140$ g/cm² of lead within the chamber geometry, is given [5] by

$$P=\frac{(Mc^2)^\alpha R_0^{1-\alpha}}{B(1-\alpha)(2-\alpha)\varrho c\tau_0}\,g\,,$$

where Mc^2 is the rest mass of the S-particle, R_0 is the vertical thickness of the plate in g/cm², B and α are the constants in the empirical range momentum relation $(P/Mc)=B(R/Mc^2)^\alpha$, ϱ is the density of the material of the plate, τ_0 is the mean lifetime of the particles, and g is the geometrical factor for our chamber. The above probability is evaluated as 10^{-3}. However, for the event in Fig. 3, this probability would be still smaller, since the secondary is emitted in the backward direction and so the S-particle has to scatter through a large angle in the plate before decaying in flight.

The two events cannot be interpreted as the nuclear captures of known negative K-mesons according to the reaction

$$K^-+N\to\Lambda^0+\pi^-+Q\,,$$

since the maximum range of the π^- in the above reaction is only 115 g/cm² of lead, taking into account the Fermi momentum of the nucleon. If one sets aside for the moment theoretical considerations [6] which demand associated production of the strange particles and assumes the following capture process, namely,

$$K^-+N\to N+\pi^-+Q\,,$$

one can explain them as unusual types of nuclear captures of the known K^--mesons. However, if one insists on the

(3) H. S. Bridge, H. Courant, H. DeStaebler and B. Rossi: *Phys. Rev.*, **95**, 1101 (1954); E. Amaldi, C. Castagnoli, G. Cortini, C. Franzinetti and A. Manfredini: *Nuovo Cimento*, **1**, 492 (1955); O. Chamberlain, E. Segrè, C. Wiegand and T. Ypsilantis: *Phys. Rev.*, **100**, 947 (1955).

(4) L. W. Alvarez, F. S. Crawford, M. L. Good and M. L. Stevenson: *Phys. Rev.*, **101**, 503 (1956); V. Fitch and R. Motley: *Phys. Rev.*, **101**, 496 (1956).

(5) H. DeStaebler jr.: *M.I.T. Thesis* (1954) (unpublished).

(6) M. Gell-Mann and A. Pais: *Proc. of the 1954 Glasgow Conference* (London, 1955).

associated production of strange particles, then for the nuclear capture process as given above, the minimum masses of these particles are 1 037 m_e and 1 125 m_e, assuming such high optimum values of Fermi energies of the capturing nucleons as 52 MeV and 70 MeV respectively.

Since we cannot exclude from ionization estimates the mass of a hyperon for the S-particle in Fig. 2, we consider the possibility of this particle being a long lived (observed time of flight 10^{-9} s), new type of hyperon, the minimum mass of which is calculated as 2 610 m_e on the basis of the decay scheme

$$Y \to N + \pi + Q\,.$$

For the S-particle in the second event in Fig. 3, the ionization estimates exclude its mass being as high as that of a hyperon.

In both these events, since we have the possibility of the secondaries escaping the chamber and thus being much more energetic than indicated by the minimum ranges given, we may discuss these events as annihilation processes of antiprotons [3,7] with nucleons in the lead nucleus. No electronic showers produced by the γ-rays from the decay of an associated π^0-meson are observed. If the π^0-meson [7] emerged in the direction opposite to the direction of the charged secondary, then in the event in Fig. 3, it would escape detection, but in the event in Fig. 2, the probability is quite high (~ 1) for detecting an electronic shower. However, the events are not inconsistent with being the annihilations of antiprotons.

We now consider the possibility of these particles being heavier than normal K-mesons ($\sim 965\ m_e$) [8], which after coming to rest in the lead plates have decayed into energetic L-mesons. We have calculated separately the minimum mass value for these two particles assuming that they decay into a μ-meson and one or more neutral particles of zero mass. The minimum masses turn out to be 1 130 m_e and 1 310 m_e for the events in Fig. 2 and Fig. 3 respectively. Assumption of any other decay scheme would yield mass values greater than the above values.

An *indication* of the existence of heavy K-mesons has been given by other workers. DANIEL *et al.* [9] of the Bristol group first reported the existence of particles of mass 1 270 m_e from direct mass measurements on shower particles from jets. Later, at the Padua Conference, FOWLER and PERKINS [10], also of the Bristol group, reported tentative results showing the existence of particles of mass 1 450 m_e. However the same authors [10] at the Pisa Conference reported that they did not find any of these particles in the G-stack; at the same time they could not explain away their earlier results in terms of known particles. HUSAIN *et al.* [11] looking for similar particles in photographic emulsions, did not find any. High $p\beta$ values for the secondaries, indicating a heavier mass for the K-mesons, have been reported by emulsion workers [12]. The cloud chamber groups at Princeton [13] and Ecole Polytechnique [14] have obtained

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(7) R. E. MARSHAK: *Meson Physics* (New York, 1952).

(8) D. M. RITSON, A. PEVSNER, S. C. FUNG, M. WIDGOFF, S. GOLDHABER and G. GOLDHABER: *Phys. Rev.*, **101**, 1085 (1956).

(9) R. R. DANIEL, J. H. DAVIES, J. H. MULVEY and D. H. PERKINS: *Phil. Mag.*, **43**, 753 (1952); R. R. DANIEL and D. H. PERKINS: *Proc. Roy. Soc.*, **221**, 351 (1954).

(10) P. H. FOWLER and D. H. PERKINS: *Suppl. Nuovo Cimento*, **12**, 236 (1954). P. H. FOWLER and D. H. PERKINS: *Proc. of Pisa Conference*, 1955 (Mimeographed report).

(11) A. HUSAIN, C. J. D. JARVIS and E. PICKUP: *Phys. Rev.*, **99**, 1612 (1955).

(12) M. G. K. MENON and C. O'CEALLAIGH: *Proc. of Bagnères Conference*, 1953 (Mimeographed report).

(13) W. H. ARNOLD, J. BALLAM and G. T. REYNOLDS: *Phys. Rev.*, **100**, 295 (1955).

(14) R. ARMENTEROS, A. ASTIER, C. D'ANDLAU, B. GREGORY, A. HENDEL, J. HENNESSY, A. LAGARRIGUE, L. LEPRINCE-RINGUET, F. MULLER, CH. PEYROU and R. RAU: *Proc. of Pisa Conference*, 1955 (Mimeographed report).

charged V-events with high momenta for the secondaries in the rest system of the primaries, thus possibly indicating higher than normal masses for the K-mesons. The Manchester group ([15]) at Jungfraujoch has observed a V^+-event which has been interpreted as being due to the decay of a particle of mass $> 1\,000\ m_e$. A positively charged V-event observed in the high pressure diffusion cloud chamber with the Cosmotron ([16]) has been interpreted as being due to the decay of a heavy K-meson giving a mass of $1\,462 \pm 15\ m_e$ with the suggested decay scheme of $K^+ \to \theta^0 + \pi^+$. The Ecole Polytechnique group ([17]) working with the double cloud chamber set up, has observed a negative particle of mass $1\,250 \pm 90\ m_e$. FRY *et al.* ([18]) studying hyperfragments have also had indications of a heavy K-meson.

The above discussion shows that the interpretation of the two unusual events as chance coincidences of unrelated tracks, or decays in flight of known K-mesons, or known hyperons, is very improbable. If they are interpreted as the nuclear capture of normal negative K-mesons, then the associated production of another «strange» particle is excluded, and we would be led to the conclusion that current theories, which require the production of such an associated particle, are incorrect. The events could be interpreted as the annihilation of antiprotons, and if this interpretation is correct, these would be the only two events of this type observed in cloud chambers so far. Finally, if the events are to be interpreted as the decay of positive K-mesons, then they would establish the existence of a new type of K-meson with a mass heavier than $1\,300\ m_e$.

* * *

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([15]) J. S. BUCHANAN, W. A. COOPER, D. D. MILLAR and J. A. NEWTH: *Phil. Mag.*, **45**, 1025 (1954).

([16]) E. M. HARTH and M. M. BLOCK: *Bull. Am. Phys. Soc.*, **30** (5), 13 (1955).

([17]) R. ARMENTEROS, B. GREGORY, A. LAGARRIGUE, L. LEPRINCE-RINGUET, F. MULLER and CH. PEYROU: *Suppl. Nuovo Cimento*, **12**, 324 (1954).

([18]) W. F. FRY, J. SCHNEPS and M. S. SWAMI: *Phys. Rev.*, **97**, 1189 (1955); W. F. FRY, J. SCHNEPS and M. S. SWAMI: *Nuovo Cimento*, **2**, 346 (1955).

IL NUOVO CIMENTO Vol. IV, N. 3 1° Settembre 1956

Perturbazione dei livelli energetici di una particella in una buca di potenziale sferoidale.

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(ricevuto il 15 Giugno 1956)

Vari Autori hanno calcolato l'energia di una particella in una scatola sferoidale, onde applicare i risultati ottenuti al modello nucleare di Rainwater [1]. È possibile generalizzare i metodi usati da FEENBERG ed HAMMACK e da GALLONE e SALVETTI, in modo da poter effettuare il calcolo perturbativo dell'energia di un nucleone in buche di potenziale sferoidali generiche, che non differiscano troppo dalla forma sferica. La presente generalizzazione implica il calcolo dei valori della funzione d'onda radiale e delle sue derivate al contorno della buca sferica imperturbata.

FEENBERG ed HAMMACK [2] considerano una buca rettangolare di profondità D, deformata come un ellissoide di rotazione. Essi danno implicitamente uno sviluppo secondo le potenze del parametro di eccentricità e, ma tale sviluppo risulta limitato in pratica al primo termine, quello lineare in e. Ora considereremo invece una buca di generico contorno a simmetria cilindrica, esprimibile nella forma:

$$s = R \tag{1}$$

ove R è il raggio della buca sferica imperturbata ed

$$s = r/\left\{1 + \sum a_n P_n(\cos\theta)\right\} = r/a\,. \tag{2}$$

Il potenziale sia:

$$V(x) = 0 \quad \text{per} \quad x > R\,; \qquad V(x) = -D \quad \text{per} \quad x < R\,. \tag{3}$$

Le autofunzioni imperturbate siano:

$$u_0(r, \theta, \varphi) = y(r) \cdot Y(\theta, \varphi)\,. \tag{4}$$

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(1) J. RAINWATER: *Phys. Rev.*, **79**, 432 (1950).
(2) E. FEENBERG and K. C. HAMMACK: *Phys. Rev.*, **81**, 285 (1951).

Applicando le regole del calcolo perturbativo del primo ordine abbiamo:

$$
\begin{aligned}
(5)\qquad E' &= \int |u_0|^2\{V(s) - V(r)\}r^2\,\mathrm{d}r \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \\
&= \int y^2(as)V(s)a^3s^2\,Y^2\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\varphi - \int y^2(r)V(r)r^2\,Y^2\,\mathrm{d}r\,\mathrm{d}\mu\,\mathrm{d}\varphi = \\
&= \int V(x)\{a^3y^2(ax) - y^2(x)\}x^2\,Y^2\,\mathrm{d}x\,\mathrm{d}\mu\,\mathrm{d}\varphi = \\
&= - D\int_0^R \{3x^2y^2 + x^3(y^2)'\}\,\mathrm{d}x\langle a - 1\rangle - \frac{D}{2}\int_0^R\{6x^2y^2 + \\
&\quad + 6x^3(y^2)' + x^4(y^2)''\}\,\mathrm{d}x\langle (a-1)^2\rangle = \\
&= - D[r^3y^2(r)]_{r=R}\langle a - 1\rangle - \frac{D}{2}\left[\{r^4y^2(r)\}' - 2r^3y^2(r)\right]_{r=R}\langle (a-1)^2\rangle ,
\end{aligned}
$$

ove:

$$
(6)\qquad \langle z\rangle = \int Y^2 z\,\mathrm{d}\mu\,\mathrm{d}\varphi .
$$

Abbiamo infine:

$$
(7)\qquad E' = - DR^3y^2(R)\sum a_nP_n(\mu) - D\{R^3y^2(R) + R^4y(R)y'(R)\}\sum a_m a_n\langle P_mP_n\rangle .
$$

Il risultato espresso dal primo termine della (7) si ottiene anche con l'altro metodo, già dato dagli Autori [3] per una buca di profilo:

$$
(8)\qquad R = R_0\{1 + \sum a_nP_n(\cos\theta)\} = R_0 + \Delta R
$$

usando quale potenziale perturbativo la espressione:

$$
(9)\qquad - D(r - R_0)\,\Delta R .
$$

Anche in questo caso, pur comprendente deformazioni più generali, i termini perturbativi considerati erano quindi quelli lineari nei parametri a_n caratteristici della deformazione.

Per il calcolo dei termini successivi, esprimiamo il potenziale nella forma:

$$
(10)\qquad V(r,\theta) = D\mathbf{1}(r - R_0) - D\delta(r - R_0)\,\Delta R + \frac{D}{2}\,\delta'(r - R_0)(\Delta R)^2 - \dots .
$$

ove

$$
(11)\qquad \mathbf{1}(x) = \frac{1}{2}\left(1 + \frac{x}{|x|}\right).
$$

[3] S. Gallone e C. Salvetti: *Phys. Rev.*, **82**, 551 (1951).

Applichiamo anche in questo vaso le regole del calcolo perturbativo del primo ordine:

$$(12)\quad \begin{aligned} E' &= -DR_0\int_0^\infty y^2(r)\,\delta(r-R_0)r^2\,\mathrm{d}r\,\langle \Delta R/R_0\rangle - \frac{D}{2}R_0^2\int_0^\infty y^2(r)\,\delta'(r-R_0)r^2\,\mathrm{d}r\,\langle(\Delta R/R_0)^2\rangle = \\ &= -DR_0^3y^2(R_0)\sum a_n\langle P_n\rangle - D\{R_0^3y^2(R_0)+R_0^4y(R_0)y'(R_0)\}\sum a_m a_n\langle P_m P_n\rangle\,. \end{aligned}$$

Se confrontiamo questa relazione con la (7), possiamo facilmente verificare la completa coincidenza dei risultati ottenuti secondo i due metodi.

A titolo d'esempio calcoleremo fino ai termini quadratici l'energia di perturbazione E', quando il profilo della buca assume la forma:

$$(13)\qquad R = R_0\{1+a_0+a_2P_2(\mu)\}\,.$$

con la condizione che il volume non dipenda dalla deformazione, condizione espressa dalla relazione:

$$(14)\qquad a_0 = -a_2^2/5\,.$$

Abbiamo:

$$(15)\quad \begin{aligned} E' &= -DR_0^3y^2(R_0)\{a_0+a_2\langle P_2\rangle\} - D\{R_0^3y^2(R_0)+R_0^4y(R_0)y'(R_0)\}\cdot \\ &\quad\cdot\{a_0^2+2a_0a_2\langle P_2\rangle + a_2^2\langle P_2^2\rangle\} = \\ &= -DR_0^3y^2(R_0)\{(a_2+2a_2^2/7)\langle P_2\rangle + 18a_2^2\langle P_4\rangle/35\} - \\ &\quad - DR_0^3y(R_0)y'(R_0)\left\{\frac{18}{35}a_2^2\langle P_4\rangle + \frac{2}{7}a_2^2\langle P_2\rangle + \frac{1}{5}a_2^2\right\}. \end{aligned}$$

È opportuno ricordare che:

$$(16)\quad \langle P_2\rangle = \frac{l(l+1)-3m^2}{(2l-1)(2l+3)}\,;\qquad \langle P_4\rangle = \frac{3}{4}\,\frac{35m^4+5\{5-6l(l+1)\}m^2+3l(l^2-1)(l+2)}{(2l-3)(2l-1)(2l+3)(2l+5)}$$

e che:

$$(17a)\qquad DR_0^3y^2(R_0) = -\frac{2(D-W)S^2_{l+\frac12}(2\sqrt{2MW/\hbar^2})}{S_{l-\frac12}(2\sqrt{2MW/\hbar^2})S_{l+\frac32}(2\sqrt{2MW/\hbar^2})}\,.$$

$$(17b)\qquad DR_0^4y(R_0)y'(R_0) = -DR_0^3y^2(R_0)\left\{1+R\sqrt{2MW/\hbar^2}+\frac{l(l+1)}{l+1+2R\sqrt{2MW/\hbar^2}}\right\},$$

ove W (positivo) $= -E$ e le S sono definite da:

$$(18)\qquad S_{l+\frac12}(2k) = i^{l+\frac32}H^{(1)}_{l+\frac12}(ik)e^k\sqrt{\pi k/2}\quad(^5).$$

(4) S. A. Moszkowski e C. H. Townes: *Phys. Rev.*, **93**, 306 (1954).
(5) E. Feenberg: *Shell Theory of the the Nucleus* (Princeton, 1955).

Al limite per una buca molto profonda si ottiene:

$$(19)\qquad DR_0^3 y_{nl}^2(R_0) \sim \frac{\hbar^2 x_{nl}^2}{MR_0^2} = 2E_{nl}\,; \qquad DR_0^4 y(R_0) y'(R_0) \sim \quad 2E_{nl}\left\{1 \quad \sqrt{\frac{2M(D-W)}{\hbar^2}}\right\},$$

ove x_{nl} è lo zero d'ordine n della funzione di Bessel di ordine $l+1/2$ ed E_{nl} è la relativa energia, nella buca sferica infinitamente profonda.

Il risultato espresso dalla (15) si può interpretare come variazione dell'energia dei nucleoni dovuta alla deformazione del nucleo. Tale variazione energetica è ottenuta in questo caso, da un calcolo perturbativo del primo ordine; il calcolo dei termini perturbativi del secondo ordine (elementi non diagonali) non è ancora stato fatto.

Nella presente approssimazione i termini prevalenti contengono a fattore gli integrali P_n, che godono della proprietà di rinormalizzare a zero il contributo energetico dei nucleoni di ogni shell satura. È su questa proprietà che si basa la possibilità di trascurare l'individualità di tutti i nucleoni appartenenti a shells sature e considerare il loro contributo all'energia di deformazione come nel modello a goccia liquida. Ricordiamo infine che MOSZKOWSKI e TOWNES [4] hanno studiato, usando relazioni lievemente diverse da quelle sopra esposte, la forma di equilibrio dei nuclei in funzione del grado di riempimento dell'utima shell, arrivando a risultati in accordo con i dati sperimentali.

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Anisotropy of Cosmic Rays due to Galactic Rotation.

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(ricevuto il 25 Giugno 1956)

1. - Several observations on the anisotropy of high energy ($\geqslant 10^{14}$ eV) cosmic ray particles have been reported in recent years (1-6). The variation observed is very small, and even the most energetic particles show a remarkable degree of isotropy. The most extensive measurements (over 4 years) of Daudin *et al.* (6) give a sidereal daily variation with an amplitude of $0.08 \div 0.16\%$, the maximum occuring at 21 h LST. It will be shown that this variation is in excellent agreement with that due to the rotation of the galaxy, the so-called Compton-Getting effect (7). Only further experiments will show whether or not this small variation and its phase are really significant, since the analysis of the experiments require a careful procedure in order to eliminate other factors which may give rise to apparent sidereal daily variations (8,9). At any rate it may be of interest, at least for future experiments, to have the results of corrected and extended calculations of the Compton-Getting effect on hand.

2. – Doppler effect and other astronomical measurements indicate that the rotational motion of our portion of the galaxy is in the galactic plane with a speed of 300 ∓ 30 km/s. The direction of the motion is towards the constellation Cygnus at $\gamma_0 = 20$ h 40 min right ascension and $\delta_0 = 47°$ N (10). The motion of the remote galaxies is relatively small, of the order of 80 km/s, so that the relative speed with respect to the sources is only approximately known, that is $\beta c = 300 \mp 100$ km/s.

(*) National Research Council Postdoctorate Fellow.

(1) A. L. Hodson: *Proc. Phys. Soc.*, **64**A, 106 (1951).

(2) A. Citron: *Zeits. f. Naturfor.*, **7**, 712 (1952).

(3) T. E. Cranshaw and W. Galbraith: *Phil. Mag.*, **45**, 1109 (1954).

(4) F. J. M. Farley and J. R. Storey: *Proc. Phys. Soc.*, **67**A, 996 (1954).

(5) G. Cocconi: *Phys. Rev.*, **83**, 1193 (1951).

(6) J. Daudin, P. Auger, A. Cachon and A. Daudin: *Nuovo Cimento*, **3**, 1017 (1956).

(7) A. H. Compton and I. A. Getting: *Phys. Rev.*, **47**, 817 (1935).

(8) H. Elliot's article in J. G. Wilson: *Progress in Cosmic Ray Physics*, vol. 1 (Amsterdam, 1952), p.484.

(9) K. Sitte: *Nuovo Cimento*, **3**, 1145 (1956).

(10) J. H. Oort: *Bull. Astr. Inst. Netherlands*, **6**, 155 (1931).

A unit area perpendicular to the motion of the earth moving with a velocity βc will receive $1/(1-(\beta/\gamma)\cos\theta)^3$ times more particles coming in straight lines with the velocity $\gamma c \approx c$ in the solid angle $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\eta$. θ is the angle between the direction of the motion and the trajectory of particles.

Let $f(E)\,\mathrm{d}E$ denote the number of particles moving towards the earth per unit solid angle in the energy range between E and $E+\mathrm{d}E$; ζ and η be the zenith and azimuth angles respectively of the incoming particles with respect to the zenith of the observer. The measured intensity at the moving system of the earth will be (1,11)

$$I' = \frac{1}{2\pi}\int_{E_0}^{\infty}\int_{\Omega}\frac{f(E)\,\mathrm{d}E}{(1-\beta\cos\theta)^3}\sin\zeta\,\mathrm{d}\zeta\,\mathrm{d}\eta\,, \tag{1}$$

where E_0 and Ω are the minimum energy and total angle of the instrument, respectively.

If I is the intensity in the rest system then the relative change, for small β, is

$$\Delta I = \frac{I'-I}{I} \simeq 3\beta\,\frac{\iint\cos\theta f(E)\sin\zeta\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}E}{\iint f(E)\sin\zeta\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}E}\,. \tag{2}$$

In particular, if the intensity is measured in a small solid angle interval, then

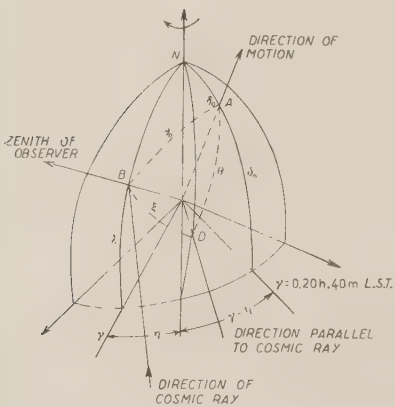

Fig. 1.

$$\Delta I(\zeta,\eta,\lambda) \simeq 3\beta\,\frac{\int_{E_0}^{\infty}\cos\theta f(E)\,\mathrm{d}E}{\int_{E_0}^{\infty} f(E)\,\mathrm{d}E}\,. \tag{3}$$

θ in general depends on ζ, η, γ, λ and E.

3. – For high energy particles we assume the trajectories to be straight lines. For an observer at geomagnetic latitude λ and longitude γ the angle θ is then easily determined (Fig. 1), and does not depend on energy E. For particles incident in the zenith direction the spherical triangle ABN gives

$$\cos\theta_0 = \sin\lambda\sin\delta_0 + \cos\lambda\cos\delta_0\cos\gamma\,, \tag{4}$$

where δ_0 is the direction of the relative motion of the earth, $\delta_0 = +47^\circ$. In the general case, $\cos\theta$ is obtained by eliminating χ from the following two equations (triangles AND and DNB)

$$\begin{cases}\cos\zeta = \sin\lambda\cos\chi + \cos\lambda\sin\chi\cos\eta\,,\\[2ex] \cos\theta = \cos\chi\sin\lambda + \sin\chi\cos\lambda\cos(\gamma-\eta)\,.\end{cases} \tag{5}$$

(11) M. S. Vallarta, C. Graef and S. Kusaka: *Phys. Rev.*, **55**, 1 (1939).

According to (3) the change of intensity for almost vertically incident particles is given by

$$\Delta I(\lambda) \simeq 3\beta\,(\sin\lambda \sin\delta_0 + \cos\lambda \cos\delta_0 \cos\gamma)\,. \tag{6}$$

In southern hemisphere λ will be taken negative. The maximum of ΔI occurs at $\gamma = 0$, that is at 20 h 40 min for all latitudes, the amplitude of the variation which is unsymmetrical depends, however, on the latitude λ:

	ΔI_{max} (%)	ΔI_{min} (%)	γ_{max}
$\lambda = 0^\circ$	$+0.16 \mp 0.05$	$-0.16 + 0.05$	20 h 40 min
$\lambda \simeq 45^\circ$	0.16 ∓ 0.05	0	20 h 40 min
$\lambda = 90^\circ$	0	0	

If the measurements cover a larger solid angle Ω' then one has to perform by means of (6) the integral

$$\Delta I = \frac{3\beta}{2\pi}\int_{\Omega'} \cos\theta \sin\zeta \, d\zeta \, d\eta\,,$$

the order of magnitude of ΔI will not change appreciably.

4. – For low energy particles the interpretation of the experiments on the sidereal variations is based on calculations by VALLARTA, GRAEF and KUSAKA [11] for particles moving in the geomagnetic equator plane of the earth. These authors have taken the direction of the linear motion of the earth due to the galactic rotation to be in the equatorial plane whereas it is directed towards $\gamma_0 = 20$ h 40 min at $\delta_0 = 47^\circ$ N. For completeness we give also the results for this case, although the effect probably does not exist for low energy particles, or would be easily masked by galactic magnetic fields.

For non-periodic orbits the angle θ is that between the direction of the relative motion of the earth and the asymptotic direction to the trajectory at infinity, since the time spent by the particle within the magnetic field of the earth is small compared with the rest of its life. Here θ clearly depends on the energy of the particles. Fig. 2 shows the angle θ for particles moving in the equatorial plane. The deflection angle χ (measured positive in the clockwise direction from the asymptote to the direction of incidence) of the trajectory is a function of E and of the angle of arrival only and has been calculated by VALLARTA *et al.* [10]. From the spherical triangle AND it follows

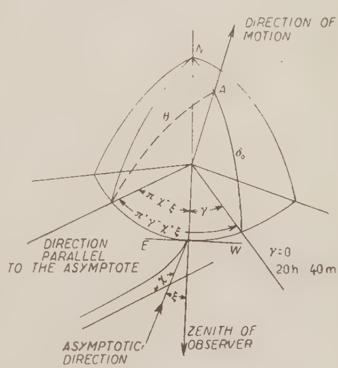

Fig. 2.

$$\cos\theta = \cos\delta_0 \cos(\pi + \gamma - \chi - \zeta)\,. \tag{8}$$

From equations (4) and (8) one obtains

$$\Delta I = -\frac{3\beta\cos\delta_0}{I}\left[(\cos\zeta\cos\gamma + \sin\zeta\sin\gamma)\int_{E_0}^{\infty}\cos\chi(E)f(E)\,\mathrm{d}E + \right.$$

$$\left. + (\cos\zeta\sin\gamma + \sin\zeta\cos\gamma)\int_{E_0}^{\infty}\sin\chi(E)f(E)\,\mathrm{d}E\right] =$$

$$= -(A_1 + B_2)\cos\gamma - (A_2 + B_1)\sin\gamma\,.$$

The maximum and the minimum of the variation occurs at γ_m which is given by

$$\text{(9)} \qquad \operatorname{tg}\gamma_m = \frac{A_2 + B_1}{A_1 + B_2} = \frac{\sin\zeta\int\cos\chi f(E)\,\mathrm{d}E + \cos\zeta\int\sin\chi f(E)\,\mathrm{d}E}{\cos\zeta\int\cos\chi f(E)\,\mathrm{d}E + \sin\zeta\int\sin\chi f(E)\,\mathrm{d}E},$$

and

$$\Delta I_{\mp} = \mp\left[(A_1 + B_2)^2 + (A_2 + B_1)^2\right]^{\frac{1}{2}}.$$

Even for an integrating chamber there is a definite time of maximum. The integrals in (9) have been approximately calculated graphically. As the primary energy spectrum that found by NEHER ([12]) has been used

$$\text{(10)} \qquad f(E) = 0.048/(E^{\frac{2}{3}} + (1 + 0.09E^{\frac{4}{3}}))^{\frac{3}{2}}.$$

For the measurement not at the top of the atmosphere one has to take into account the response of the particle detector. FONGER ([13]) has given the effective primary proton spectra for ionisation chambers and neutron detectors. For the equator plane where the low energy cut off for protons is about 14 GeV the effective spectrum is very similar to the primary spectrum and would change the results only slightly. However, this may be important at higher latitudes.

Some approximate results obtained on this basis are as follows:

		$\Delta I_{\max}$ (%)	γ_m (LST)
$\lambda = 0$	$\zeta = +30^\circ$ (East incidence)	$0.075 + 0.025$	12 h
$\lambda = 0$	$\zeta = 0$ (Vertical incidence)	$0.105 + 0.035$	15 h
$\lambda = 0$	$\zeta = -30^\circ$ (West incidence)	$0.12 + 0.04$	17 h

The variation is the greater the greater the total flux of cosmic ray particles. Therefore due to the geomagnetic cut-off we would expect a higher $\Delta I_{\max}$ at higher latitudes λ. On the other hand there will be a decrease of $\Delta I_{\max}$ due to the change of $\cos\theta$ (the variation is zero at the poles) so that the same order of magnitude for $\Delta I_{\max}$ can be estimated for $\lambda \approx 45^\circ$.

([12]) H. V. NEHER: *Phys. Rev.*, **83**, 649 (1951).
([13]) W. H. FONGER: *Phys. Rev.*, **91**, 351 (1953), Fig. 2.

5. – For high energy particles the agreement between the observed values by DAUDIN *et al.* ($\lambda = 46^{\circ}$ N) and that calculated on the basis of an isotropic extragalactic distribution is very satisfactory. Two observations [1,4] report a higher amplitude, 1.15% and 1.43% respectively; in all other cases the expected variation is within the experimental error.

The experiments for the low energy part of the spectrum have been reviewed and analysed by ELLIOT [8]. ELLIOT and DOLBEAR have found that the mean sidereal variation of the total cosmic radiation has an amplitude of about 0.022% and the time of maximum is at about 3.5 h LST.

If further experiments confirm the agreement reported for the anisotropy of high energy particles, then this would suggest that the high energy primaries have extragalactic origin, for it appears that other galaxies are distributed isotropically around our own galaxy. Any point source of high energy cosmic rays would produce a higher anisotropy. Furthermore one would have to make very special hypotheses regarding the galactic fields and the mechanism of acceleration to explain the low degree of anisotropy of high energy particles within the galaxy, and the agreement is less good [14].

The main difficulty of an extragalactic origin of all the cosmic radiation is that the energy in the form of cosmic rays would be about the same as that in the form of starlight. This may not be improbable for high energy particles only, for the total number of cosmic ray particles, near the earth, with energies greater than 10^{14} eV is about 10^{-7} times the total cosmic ray flux. Furthermore it may be that the high energy particles have a non-thermodynamic origin. Another support to the present conclusion comes from the fact that the cosmic ray particles may escape the galaxy [15] and this is probably true for other galaxies too.

Perhaps the final answer to this question will be brought when we know more about the galactic fields and acceleration mechanism of cosmic rays.

* * *

It is a pleasure to thank Professor J. A. SIMPSON for his suggestion of the problem and for helpful discussions.

Note added in proof: In the meantime G. COCCONI has given additional arguments for the extragalactic origin of the most energetic cosmic ray particles: *Nuovo Cimento*, **3**, 1433 (1956).

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(14) L. DAVIS: *Phys. Rev.*, **96**, 743 (1954).
(15) P. MORRISSON, S. OLBERT and B. ROSSI: *Phys. Rev.*, **94**, 440 (1954).

Depolarization of Stopped Particles.

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(ricevuto il 5 Luglio 1956)

A previously ([1]) suggested search for angular correlations in τ-meson decay has shown a negative result. Since by now the lifetime of the τ-meson is known ([2]) to be of the order of $2\cdot 10^{-8}$ s one might ask whether this negative result means the absence of an initial polarization or that the τ-meson becomes depolarized. Since the answer to this question depends on the magnetic moment of the meson and furthermore the question about the spin of heavy mesons is still open it seems of some interest to discuss this point a little further. Experimentally nothing is known about the electric quadrupole moments of elementary particles. Theoretically one would expect them to be of the order $Q \sim e(\hbar/Mc)^2$. If this is true the dominant depolarizing factor will be magnetic interactions since their interaction energy is of the order $E_{\text{m.d.}} \sim H\mu_{\text{mes}} \sim m_e c^2\alpha^4 m_e/M$, whereas the quadrupole interaction is only $E_{\text{e.q.}} \sim (Q, \text{grad } E) \sim m_e c^2\alpha^4(m_e/M)^2 = 10^{-3}E_{\text{m.d.}}$. During the slowing down the meson experiences rapidly (10^{-13} s) fluctuating fields and they will generally not depolarize the meson. We shall see that only magnetic fields acting for a long time in the same direction (10^{-8} s) will turn the spin around. The relevant quantity is the Larmor frequency with which the spin of the meson will precess during its magnetic interaction with an electron. It is given by the well known ([3]) formula

$$\omega = \frac{2\pi c R\alpha^2(F(F+1) - I(I+1) - j(j+1))}{n^3(2l+1)j(j+1)}\,\mu\,. \tag{1}$$

This frequency is for normal values of the magnetic moment of the meson μ less than the inverse lifetime due to optical ([4]) transitions and the decay of the τ-meson except for the $n=1$ state. In the following table we list for the first two states the inverse average values for the frequency differences and the lifetimes in those states.

([1]) M. Teucher, W. Thirring and H. Winzeler: *Nuovo Cimento*, **1**, 733 (1955).
([2]) L. Alvarez *et al.*: *Phys. Rev.*, **101**, 503 (1956).
([3]) H. Kopfermann: *Kernmomente*, p. 109 (1956).
([4]) H. A. Bethe: *Handbuch der Physics*, **23**$_1$, p. 444 (1933).

We shall discuss only $I=1$, mesons with higher spins give about the same values as here but will be much less depolarized as shown below.

$n=l+1$	$(\overline{\Delta\omega})^{-1}$	$\tau=(\tau_{\text{el.}}^{-1}+\tau_{\text{dec}}^{-1})^{-1}$
1	$1.5\gamma\cdot 10^{-9}$ s	10^{-8} s
2	$5.4\gamma\cdot 10^{-8}$ s	$1.4\cdot 10^{-9}$ s

$(\gamma=\hbar e/2mc\mu)$

In the $n=1$ state the electron has angular momentum $\frac{1}{2}$ and will, therefore, little affect the polarization of the meson unless its spin is low. To illustrate this we shall calculate the change of the angular correlations of the decay of a particle with spin 1 and 2 into 2 spinless particles (due to its coupling with the electron). The state vector for the two possible spin orientations of the electron will be of the form

$$(2)\qquad \begin{cases} \psi\uparrow\uparrow(t)=|I+\frac{1}{2},\, I+\frac{1}{2})\exp[it\omega_1]\,, \\ \psi\uparrow\downarrow(t)=\left(\dfrac{1}{2I+1}\right)^{\frac{1}{2}}|I+\frac{1}{2},\, I-\frac{1}{2})\exp[it\omega_1]+\left(\dfrac{2I}{2I+1}\right)^{\frac{1}{2}}|I-\frac{1}{2},\, I-\frac{1}{2})\exp[it\omega_2]\,. \end{cases}$$

For the distribution function

$$(3)\qquad D_I(\vartheta)=\frac{1}{\tau}\int_0^\infty \mathrm{d}t\exp[-t/\tau]\tfrac{1}{2}(|\psi\uparrow\uparrow(t)|^2+|\psi\uparrow\downarrow(t)|^2),$$

we find

$$D_1(\vartheta)=\frac{1}{48\pi}\left[\sin^2\vartheta\left(14+\frac{4}{1+\tau^2(\Delta\omega)^2}\right)+\cos^2\vartheta\left(8-\frac{8}{1+\tau^2(\Delta\omega)^2}\right)\right],$$

$$D_2(\vartheta)=\frac{3\sin^2\vartheta}{160\pi}\left[\sin^2\vartheta\left(21+\frac{4}{1+\tau^2(\Delta\omega)^2}\right)+\cos^2\vartheta\left(16-\frac{16}{1+\tau^2(\Delta\omega)^2}\right)\right].$$

ϑ is the angle between the spin direction and the direction of the decay. In the two limits we get

$$(4)\qquad \begin{cases} & \tau\ll\omega^{-1} & \tau\gg\omega^{-1} \\ D_1 & \sin^2\vartheta & 1+\frac{3}{4}\sin\vartheta \\ D_2 & \sin^4\vartheta & \sin^2\vartheta\left(1+\dfrac{5}{16}\sin^2\vartheta\right). \end{cases}$$

Consequently the angular correlations will be somewhat quenched in the spin 1 case if an electron reaches the K-orbit soon enough. Whether this happens will

depend on the substance. In AgBr the distance between two atoms is about 10 Bohr radii so that some electrons will go into $n \geqslant 3$ states which have a lifetime $> 10^{-8}$ s and will, therefore, not reach the ground state. For spin > 1 the angular correlations will persist unless the meson has an exceedingly large magnetic moment ($\gamma^{-1} \geqslant 50$) which makes the interaction with the $p_{\frac{3}{2}}$ electrons strong enough.

* * *

I would like to thank Prof. M. FIERZ for a stimulating discussion.

ERRATA-CORRIGE

W. CZYŻ and J. SAWICKI **Polarization of Nucleons from Photonuclear Reactions:** *Nuovo Cimento* **3**, 864-869 (1956).

	instead of:	should read:
p. 865, line 2, 3 from the top	$\hat{I}^2$	$\hat{J}^2$
p. 867, line 6 from the top		$I = \frac{4}{3}\, R_{\frac{1}{2}} + \frac{1}{10} R_{\frac{3}{2}} + \frac{9}{10} R_{\frac{5}{2}} \Big|^2 + \frac{9}{25} \Big| R_{\frac{3}{2}} - R_{\frac{5}{2}}$
p. 867, line 8 from the top	$C = R_{\frac{1}{2}} + \frac{5}{2} R_{\frac{3}{2}} - \frac{9}{10} R_{\frac{5}{2}}$	$C = R_{\frac{1}{2}} + \frac{2}{5} R_{\frac{3}{2}} \quad \frac{9}{10} R_{\frac{3}{2}}$
p. 867, line 11 from the bottom	$= 1.6$ MeV	$= 1.63$ MeV (according to [3]). However, the last experimental value is 1.66 MeV. This difference has little influence on the numerical results.
p. 868, line 8-11 from the top	This result .... Austern [5]	This result may, for instance, be applied to the calculation of polarization of photoneutrons from the ^{29}Si nucleus close to the threshold of the ^{29}Si(γ, n) reaction.

A Test for the "Median Angle,, Method (*).

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(ricevuto il 14 Luglio 1956)

For several years the median angle method has been used for the determination of the energy of particles producing stars in photographic emulsions. It has been extensively discussed in the literature [1-5]. The purpose of this note is to report on the application of this method to stars produced by protons from the Berkeley Bevatron, and thereby test the procedure at one energy, namely 6.2 GeV (6.6 Mc^2). The stars were grouped according to the number of heavy and grey prongs N_h. The light tracks ($g \leqslant 1.55 g_{\min}$) of the stars within each group were added up to form a « composite star » characterized by the common value of N_h. The results are shown in Table I. n_s is the total number of light prongs in each « composite star. » γ_c is calculated according to CASTAGNOLI *et al.* [3] $\log \gamma_c = -1/n_s \sum_i \log \operatorname{tg} \theta_i$. E is the apparent kinetic energy in the laboratory system in Mc^2. For comparison, the γ's have also been evaluated directly by observing the median angle in each group. The results are close and run parallel to those given in the Table.

TABLE I.

N_h	n_s	γ_c	E
0	15	2.44	$10.8^{+6.8}_{-4.6}$
2	12	7.1	98^{+70}_{-40}
3	31	3.1	$17.4^{+7.4}_{-5.4}$
4	21	2.1	$6.8^{+4.4}_{-2.9}$
5	19	1.76	$4.2^{+3.2}_{-2.1}$
6	39	3.0	$16.9^{+6.0}_{-2.4}$
7	13	2.6	$11.0^{+10.0}_{-5.4}$
8	15	1.92	$5.4^{+4.4}_{-2.8}$
9	16	1.83	$4.6^{+4.0}_{-2.6}$

(*) This work was supported by the joint program of the ONR and AEC.

(1) H. L. BRADT, M. F. KAPLON and B. PETERS: *Helv. Phys, Acta*, **23**, 24 (1950).
(2) C. C. DILWORTH, S. J. GOLDSACK, T. F. HOANG and L. SCARSI: *Nuovo Cimento*, **10**, 1201 (1953).
(3) C. CASTAGNOLI, G. CORTINI, C. FRANZINETTI, A. MANFREDINI and A. MORENO: *Nuovo Cimento*, **10**, 1539 (1953).
(4) G. BERTOLINO and D. PESCETTI: *Nuovo Cimento*, **12**, 630 (1954).
(5) A. ENGLER, U. HABER-SCHAIM and W. WINKLER: *Nuovo Cimento*, **12**, 930 (1954).

The following conclusions may be drawn from these results.

1) Energy determinations are subject to very large fluctuation.

2) Contrary to common belief, the derived value of the energy is much more likely to be too large than too small. This seems to be the case even for stars with $N_h=7$. Adding all stars with $0 \leqslant N_h \leqslant 7$ together we find $E=11.7 \pm 2.2$. Several reasons may be responsible for this result. At 6.2 GeV the average number of charged pions produced in a collision is quite small, perhaps $2 \div 3$, and therefore a sizable fraction of the light tracks will be due to protons. These obviously do not fulfill the conditions necessary for the application of the median angle method; namely, that their velocity in the c.m. system will be at least equal to that of the primary proton in that system. This will cause a collimation in the forward direction. The situation is not improved by using a « composite star » since such a « star » contains the same percentage of protons as the average individual star. In addition to the secondary protons which are inherently too slow, a certain fraction of the pions may be produced with an energy less than 140 MeV in the center of mass system and those too will be collimated in the forward direction more than they should be, since also their velocity will be smaller than that of the primary protons. Finally a fraction of the pions can be quite energetic in the center of mass system yet be slow enough in the laboratory system to produce a gray track which will not be counted. Evaluating this effect requires detailed assumption on the energy and angular distributions of the pions in the center of mass system. The precise cut-off value for the ionization $g=1.55g_{\min}$ is not of great importance since there were only a few border line tracks.

It is difficult to extrapolate these results, obtained at 6.2 GeV, to the very high energies found in cosmic rays. But one should perhaps observe some caution when applying the median angle method to intermediate energies of say, $10 \div 30$ GeV.

* * *

It is a pleasure to thank Professor R. D. Hill for making his developed plates available for this experiment and Professor G. Ascoli for several stimulating discussions.

IL NUOVO CIMENTO Vol. IV, N. 3 1° Settembre 1956

Measurement of the Energy Spectrum of Protons from (n, p) Reactions on Mg and Zn with 14 MeV Neutrons.

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CISE - Milano

(ricevuto il 16 Luglio 1956)

A considerable deal of attention has been paid in the last few years to nucleons emitted by nuclei excited at energies up to about twenty MeV, in order to test some basic assumptions in nuclear dynamics, like the compound nucleus formation and the statistical evaporation theory. Some experimental results were not found in good agreement with these assumptions in the case of (p, p′) anelastic scattering and (γ, p) and (n, p) reaction cross-sections [1] and therefore new mechanisms were proposed in which one or more nucleons of the bombarded nucleus were excited (direct interactions) [2].

The measurement of the energy spectrum of protons emitted in (n, p) reactions with nuclei of medium atomic weight bombarded with 14 MeV neutrons is a good source of information about the mechanism of these reactions and can particularly prove the validity of the evaporation model.

So far only in the case of the reaction Cu (n, p) has a measurement of this spectrum been published (by D. L. Allan [3]) and was obtained by nuclear plates technique.

The purpose of the present work is to show some preliminary results we obtained with a special arrangement of a proportional counter and a scintillating crystal which are represented in Fig. 1.

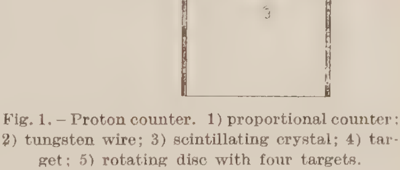

Fig. 1. – Proton counter. 1) proportional counter; 2) tungsten wire; 3) scintillating crystal; 4) target; 5) rotating disc with four targets.

Inside the proportional counter at one side is placed the target in which the

(1) D. C. Peaslee: *Ann. Rev. of Nuclear Science*, **5**, 99 (1955).

(2) *Statistical aspects of the Nucleus* - Report on Brookhaven Conference, Upton, N.Y., BNL 331 (1955).

(3) D. L. Allan: *Proc. Phys. Soc.*, A **68**, 925 (1955).

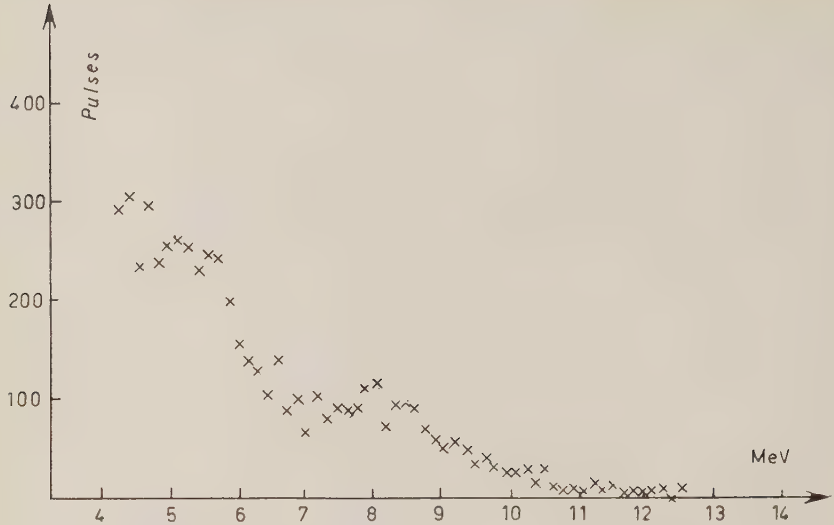

Fig. 2. – Energy spectrum of protons emitted in reaction Mg(n, p) by 14 MeV neutrons. Statistical errors: ± 10%. A broad bump is shown at ∼ 8 MeV proton energy.

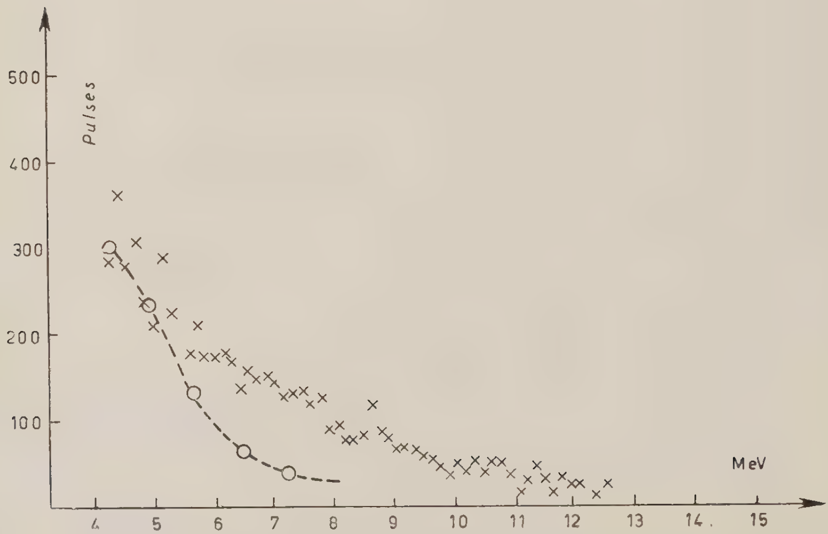

Fig. 3. – Energy spectrum of protons emitted in reaction Zn(n, p) by 14 MeV neutrons. Statistical errors: ± 10%. The dashed curve is calculated with the evaporaton model of Weisskopf.

(n, p) reactions are produced and at the other side a scintillating crystal thick enough to stop protons of the maximum expected energy. A rotating support is provided in order to change easily the target and to measure the background of the apparatus by means of a thick target of carbon or lead which practically does not produce any (n, p) reactions.

The counter is filled with 20 cm_{Hg} of an argon-nitrogen mixture. The scintillating crystal is viewed by a photomultiplier connected with a proportional electronic chain; the pulses from the proportional counter open an electronic gate through which the pulses from photomultiplier, when in coincidence, pass and are sent to a 99 channels pulse analyzer. The bias of the proportional counter can be regulated to discriminate proton pulses from electrons.

The assembly is placed at a distance of a few centimetres from the target of a Cockcroft Walton accelerator operated with a $d+t$ reaction and produces a total flux of $\sim 10^7$ neutrons-second of 14 MeV energy.

With these techniques it has been possible to observe the protons emitted in (n, p) reactions in the case of many nuclei of medium weight used as targets: magnesium, aluminum, silicon, sulfur, copper and zinc.

Preliminary results on the energy distribution of the protons in the cases of Mg and Zn are shown in Figg. 2 and 3.

The thickness of targets is between 5 and 12 mg/cm^2; the correction for the self absorption in the target and for the absorption through all the instrument ranges from 0.5 MeV to 0.3 MeV for energies of the protons between 4 and 14 MeV.

The angle of acceptance of the protons has been evaluated by studying the recoil protons from a hydrogen target and found to be between 0 and 60 degrees in the forward direction by means of the same measurements it has been possible to establish the energy scale which is given with the approximation of about 10% and the energy resolution, evaluated at $\sim 10\%$.

Our results do not show pure evaporation spectra, particularly in the case of Mg: there are many more protons in the high energy regions than the evaporation itself should supply.

These results seem to evidence the presence of a direct interaction mechanism: in fact in such processes the shape of the spectrum is expected to be rich with high energy protons.

Further results of D. L. ALLAN [4] obtained with nuclear plates indicate similar deviations from the evaporation shape in reactions Fe (n, p) and Ni (n, p).

[4] Private communication.

Nature of the λ-Transition in Liquid Helium.

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(ricevuto il 24 Luglio 1956)

Although it is generally believed that the λ-transition in liquid helium is somehow related to the condensation mechanism of an ideal Bose Einstein gas, the theories proposed so far either give a transition of the third order (continuous specific heat) ([1,2]) or else provide no mechanism for a thermodynamic transition of any kind ([3]).

We have been able to show that the reason for the third order transition obtained in references ([1]) and ([2]) is the neglect of the effects of the attractive interaction between helium atoms. Although the main interaction is of the « repulsive core » type, the weak attractive forces cannot be neglected. They are essential not only for the cohesion of the liquid but also for the understanding of the nature of the λ-transition. As a result of the attractive interactions, the very large « cycles » of the Kahn-Uhlenbeck method ([4]) tend to concentrate into smaller regions of space than they do in the ideal gas. Indeed, the average volume occupied by a cycle of size l is proportional to l itself rather than to $l^{\frac{3}{2}}$, for large values of l. The number of particles appears as a power series in the activity z; in the ideal gas, this power series converges on its radius of convergence; in the present theory, the power series diverges on its radius of convergence. In consequence, there is now *no thermodynamic transition whatever*. Rather, we get a phenomenon which we shall call a *quasi-transition*. Various thermodynamic properties change continuously and differentiably with temperature, but undergo large changes in a small temperature interval ΔT in the neighbourhood of a « quasi-transition temperature » T_0, such that $\Delta T \ll T_0$. Unlike a real thermodynamic transition, the temperature interval ΔT is independent of the volume of the container.

(*) Also supported by the Nuclear Research Foundation within the University of Sydney.

([1]) R. P. Feynman: *Phys. Rev.*, **91**, 1291 (1953).

([2]) S. T. Butler and M. H. Friedman: *Phys. Rev.*, **98**, 287, 294 (1955).

([3]) L. Landau: *Journ. of Phys.* (*U.S.S.R.*), **5**, 71 (1941).

([4]) B. Kahn and G. E. Uhlenbeck: *Physica*, **5**, 399 (1938).

We use the term « cycle » in order to avoid confusion with the cluster integrals which appear in the Mayer theory of an imperfect gas. This latter theory is of course inapplicable to a liquid, and has not been used in this investigation.

On the other hand, if ΔT is smaller than the temperature intervals resolved experimentally, the quasi-transition appears experimentally entirely similar to a true thermodynamic transition. The small value of ΔT in liquid helium is due to the fact that the very large cycles make essentially no contribution to the number of particles until z is extremely close to the radius of convergence of the power series; at that point a small change in z produces very large effects on the thermodynamic properties.

Besides the abrupt change in the specific heat c_V at the λ-point, it is necessary to explain the very sharp temperature dependence of c_V *below* the λ-point; prior theories ([1,2]) gave c_V essentially proportional to $T^{\frac{3}{2}}$. In our theory, a large cycle tends to exclude part of the volume which it occupies from occupation by « single » particles (particles not linked to other by statistical links ([2])). In the extreme limit of a tightly bunched-up cycle, the entire volume covered by the cycle is excluded; in that case the quasi-transition becomes first order, i.e., below $T_0-\Delta T$ all the particles of the liquid belong to large cycles, and the internal energy and entropy are substantially zero. In the physical case, of partial volume exclusion, the quasi-transition is of second order, and as T is lowered, a very rapidly increasing fraction of the particles belong to very large cycles. The precise fraction of the cycle volume excluded can not be calculated at this time, and has therefore been used as a (temperature-independent) parameter. The best value is close to $\frac{1}{2}$, which is entirely reasonable.

The result for the specific heat, on the basis of several crude and preliminary approximations, has been evaluated with the help of SILLIAC, the electronic computer in the School of Physics, and is shown in Fig. 1. It should be noted that c_V drops by more than a factor 2

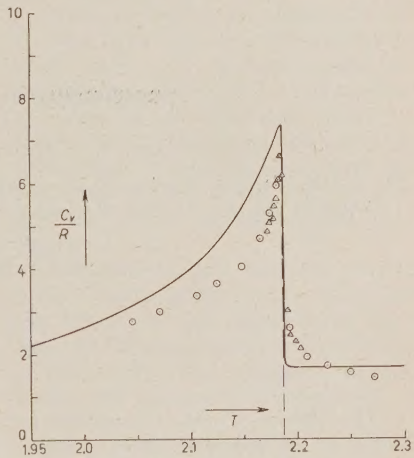

Fig. 1. – The specific heat of liquid helium vs. temperature, in the neighbourhood of the λ-point. The curve is theoretical, the experimental points are taken from KEESOM: *Helium* (Amsterdam, 1942), Table 4.20. The theoretical curve is a continuous and differentiable function of temperature everywhere, represented by the same theoretical expression throughout the temperature region covered in the figure.

in an interval of the order of 10^{-3} degrees. This is so in spite of the fact that the only temperature dependence appearing explicitly in the theoretical expressions is a very slow $T^{\frac{3}{2}}$ dependence. c_V is obtained from the theoretical formula for the free energy by two differentiations with respect to temperature, and hence only qualitative agreement can be expected from this first, crude approximation. It can be shown that various finer effects, which will be included in later work, go in the right direction to improve the agreement with experiment.

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Superfluidity of Liquid Helium.

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The theory of the λ-transition reported in the preceding letter (1) allows a rough estimate of the correlation length (2,3) Λ in liquid helium. Λ is given in order of magnitude by the linear dimensions of the contributing very large cycles. The latter are around $10^6 < l < 10^8$, giving $\Lambda \sim 10^{-5}$ cm. This estimate is consistent with the upper limit given in reference (3), which was 10^{-2} cm. Thus (3) the superfluid properties of liquid helium are non-equilibrium phenomena which cannot be logically deduced from the present theory which is restricted to thermodynamic equilibrium. Nevertheless, it is tempting to make some speculations along the following lines:

The power series for the number density of particles, $\varrho = N/V$, in powers of the activity z, splits naturally into two parts, a contribution from the very small cycles, which we shall call ϱ_n, and a contribution from the very large cycles, which we shall call ϱ_s:

$$\varrho = \varrho_n + \varrho_s \,. \tag{1}$$

Although the separation into these two contributions is not entirely sharp, it is sufficiently sharp for practical purposes, i.e., there exists an intermediate range of cycle sizes l which contributes extremely little to ϱ.

The large cycles contribute significantly to ϱ, but make negligible contributions to either the entropy, internal energy, or specific heat. It is therefore suggestive to identify ϱ_s in (1) with the « density of superfluid », ϱ_n with the « density of normal fluid », in the usual two-fluid model (4) of liquid helium.

Before such an identification can be made, however, it is necessary to understand how these very large cycles can move freely through the liquid, with nearly zero friction. The following mechanism appears plausible to us. Although a large cycle can not easily appear or disappear in its entirety, single particles may be incorporated into the cycle, and cycle particles may be removed

(*) Also supported by the Nuclear Research Foundation within the University of Sydney.

(1) S. T. BUTLER, J. M. BLATT and M. R. SCHAFROTH: *Nuovo Cimento*, **4**, 674 (1956).

(2) J. M. BLATT, S. T. BUTLER and M. R. SCHAFROTH: *Phys. Rev.*, **100**, 481 (1955).

(3) S. T. BUTLER and J. M. BLATT: *Phys. Rev.*, **100**, 495 (1955).

(4) L. LANDAU: *Journ. of Phys.* (*U.S.S.R.*), **5**, 71 (1941); L. TISZA: *Phys. Rev.*, **72**, 838 (1947).

from the cycle to become single particles again. Thus the large cycle may effectively move through the liquid by « eating its way through ». This provides a mechanism for « internal convection » of the superfluid with respect to the normal fluid, as is observed for instance in second sound ([4]).

The presently available theory of transport phenomena in statistical mechanics is not applicable to this system; hence it is difficult to make estimates of relaxation times. However, some very

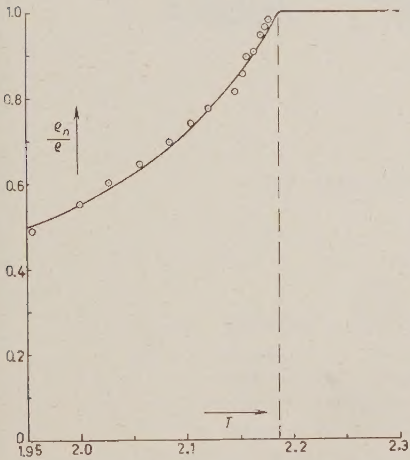

Fig. 1. – The normal fluid concentration ϱ_n/ϱ in liquid helium vs. temperature, in the neighbourhood of the λ-point. The curve is theoretical, the experimental points are based on the second sound data of PELLAM ([5]). The theoretical curve is a continuous and differentiable function of temperature everywhere, represented by the same theoretical expression throughout the temperature region covered in the figure.

crude estimates based on this qualitative picture are entirely consistent with very long relaxation times, i.e., substantially frictionless superflow.

When we identify the contribution of the very large cycles to ϱ with the superfluid density ϱ_s, it becomes possible to calculate the normal fluid concentration ϱ_n/ϱ from the thermodynamic theory ([1]). Unlike the specific heat, no differentiation with respect to temperature is involved, and hence even a rather crude theory may give sensible agreement with experiment.

The curve for ϱ_n/ϱ computed on this basis is shown in Fig. 1; also shown on the figure are the experimental data based on the second sound measurements of PELLAM ([5]). The agreement between theory and experiment is adequate.

* * *

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([5]) J. R. PELLAM: *Phys. Rev.*, **75**, 1183 (1949); **78**, 818 (1950).

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